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Proportional Reasoning

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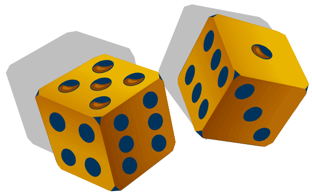
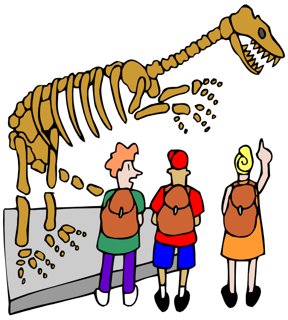
Proportional Reasoning

Introduction



Many topics within the school mathematics and science curriculum require knowledge and understanding of ratio and proportion. In mathematics, for example, problem solving and calculation activities in domains involving scale, probability, percent, rate, trigonometry, equivalence, measurement, the geometry of plane shapes, algebra are assisted through ratio and proportion knowledge. In science, calculations for density, molarity, speed and acceleration, force, require competence in ratio and proportion. As ratio and proportion permeate so many topics in mathematics and science, the importance of study of these two concepts in the school curriculum is highlighted.

In the middle years of schooling, ratio and proportion are typically studied in mathematics classes. In fact, ratio and proportion have been described as the cornerstone of middle years mathematics curriculum. However, research has consistently highlighted students' difficulties with proportion and proportion-related tasks and applications, which means that many students will struggle with topics within both the middle years mathematics and science curriculum due to their lack of understanding of ratio and proportion. Understanding ratio and proportion is more than merely being able to perform appropriate calculations and being able to apply rules and formulae, and manipulating numbers and symbols in proportion equations. Educators are well aware that students' computational performances are not a true indicator of the degree to which they understand the concepts underlying the calculations. Understanding in mathematics is generally described as a way of knowing. Knowing ratio and proportion is about proportional reasoning.



Proportional Reasoning

Before considering proportional reasoning, consider the meaning of the words ratio and proportion. In its barest form, ratio describes a situation in comparative terms, and proportion is when this comparison is used to describe a related situation in the same comparative terms. For example, if we say that the ratio of boys to girls in the class is 2 to 3, we are comparing the number of boys to the number of girls. When we know that there are 30 children in the class we know that, proportionally, the number of boys is 12 and the number of girls is 18. We are using the base comparison to apply it to the whole situation. In order to understand this relationship proportional reasoning is used. Proportional thinking and reasoning is knowing the multiplicative relationship between the base ratio and the proportional situation to which it is applied.

Proportional reasoning is being able to make comparisons between entities in multiplicative terms. This means that the relationship between the two entities is conceptualised as a multiplicative relationship. For many young children, comparisons between entities are described in additive terms, and they compare groups using additive or subtractive language. For example, when comparing the number of boys to girls as in the example above (ratio of boys to girls 2:3), they may say that there is always one extra girl for each group of boys. So, if there were 4 boys, there would be five girls. Being able to describe proportional situations using multiplicative language is an indicator of proportional reasoning.

The development of proportional reasoning is something that takes time. It is fostered by quality learning experiences in which students have opportunities to explore, discuss and experiment with proportion situations. Proportional reasoning is also dependent upon sound foundations of associated topics, particularly multiplication and division. Other mathematical topics through which proportional reasoning grows is through the study of rational number topics including fractions, decimals, percentages, scale drawing, and of course ratio and proportion. Proportional reasoning is fostered through rich conceptual understanding of ratio and proportion, but these are difficult concepts that present a challenge to both students and their teachers.

Proportional Reasoning in Science and Mathematics

Mathematics and science educators are increasingly talking about proportional reasoning as a fundamental link between mathematics and science. As outlined above, proportional reasoning is required to operate in many mathematics and science topics. Proportional reasoning is also fundamental to real-world and every-day situations, and hence underpins numeracy and scientific reasoning. For example, determining comparing 200 grams for $3 or 250 grams for $4; or understanding why a baby locked in a car on a hot day suffers more than an adult in the same circumstances, involve proportional reasoning. Drawing a plan view of a house, a “mud map” of the path from home to school, and a plan of the school yard; sharing four pizzas between three people or two chocolate bars between three people; determining the better buy when 1 kg costs $3.50 and 1.5 kg costs $4.20; determining whether there is more chance of selecting red from a collection of 3 red and 4 blue compared to a collection of 6 red and 8 blue, all require proportional reasoning. Underdeveloped proportional reasoning potentially impacts real-world situations, sometimes with life-threatening or disastrous consequences (e.g., administration of incorrect doses in medicine; inaccurate mix of chemicals in pesticides; failure to accurately convert between units of metric and imperial units of measure). A rich field of potential investigations that link science and mathematics opens up when one considers the multi-faceted and multi-dimensional nature of proportional reasoning.

In the middle years of schooling, instruction must take an active role in supporting the development of proportional reasoning. It is estimated that only 50% of the adult population can reason proportionally. The question must be asked as to why this is the case. As proportional reasoning permeates so many topics in science and mathematics, it is timely to consider our role as teachers in fostering students’ proportional reasoning skills. Proportional reasoning does not develop in a linear fashion and is something that is never an absolute state. It will continue to grow and develop with experience and explicit teaching. Mathematics and science teachers working together can mutually support each other in developing rich learning experiences that assist students to make connections between key topics across these two disciplines as well as foster numeracy and scientific thinking and reasoning.

Thinking about Proportional Reasoning

To immerse you in ratio and proportion, and to start you thinking about the range of strategies that can be applied to ratio and proportion tasks, try to do the following in your head. All these problems require proportional reasoning, and through engaging with each task, you will become more aware of the range of tasks that require proportional reasoning, and subsequently, the breadth of thinking that is incorporated within proportional reasoning. Don't be tempted to do your calculations using learned procedures, rules and algorithms on paper, and then scribbling your working out. Just try to jot down things to assist your memory without resorting to completion of pen and paper methods. Work through them on your own first before you share your thinking with others.

1. Six workers can build a house in 3 days. Assuming that all of the workers work at the same rate, how many workers would it take to build a house in 1 day?

2. Eighty M&Ms will be divided between two children in the ratio 2:3. How many M&Ms will each child receive?

3. If 5 chocolates cost $.75, how much do 13 cost?

4. Between them, John and Mark have 32 marbles. John has 3 times as many as Mark. How many marbles does each boy have?

5. Jane loves to read. She can read a chapter in about 30 minutes. Assuming chapters are all about the same length, how long will it take her to read a book with 14 chapters?

6. Six students were given 20 minutes to clean up the classroom after an eraser fight. They were angry and named 3 other accomplices. The principal added their friends to the clean-up crew and changed the time limit. How much time did she give them to complete the job?

7. If 1 football player weighs 115 kg, what is the total weight of 11 starters?

8. Sandra wants to buy an MP3 player costing $210. Her mother agreed to pay $5 for every $2 Sandra saved. How much did each pay?

9. A company usually sends 9 people to install a security system in an office building, and they do it in about 96 minutes. Today, they have only three people to do the same size job. How much time should be scheduled to complete the job?

10. A motor bike can run for 10 minutes on $.30 worth of fuel. How long could it run on $1.05 worth of fuel?

Further thinking about proportional reasoning

Here are some more problems that require proportional reasoning. They require a bit more thinking than the previous problems so don't give up if you haven't got a solution after a few minutes. Give yourself time to think over these problems and what they are asking. Leave them for a while and come back to them if you are unsure. It is amazing what the mind can achieve when it is thinking of other things.

11. On a sunny day, you and your friend were taking a long walk. You got tired and stopped near a telephone pole for a little rest, but your nervous friend couldn't stand still. He pace out your shadow and discovered it was 5 metres long even though you are really only 1.5 metres tall. He paced the long shadow of the telephone pole and found that it was 35 metres long. He wondered how high the telephone pole really is. Can you figure it out?

12. Which is more square, a rectangle that measures 35cm x 39cm or a rectangle that measures 22cm x 25cm?

13. Two gears, A and B, are arranged so that the teeth of one gear mesh with the teeth of another. Gear A turns clockwise and has 54 teeth. Gear B turns counter clockwise and has 36 teeth. If gear A makes 5.5 rotations, how many turns will gear B make?

14. Mr Brown is a bike rider. He considered living in Allentown, Binghamton, and Chester. In the end, he chose Binghamton because, as he put it, "All else being equal, I chose the town where bikes stand the greatest chance on the roads against cars." Is Binghamton town A, B, or C?

A. Area is 15 ha.; 12,560 cars in town.
B. Area is 3 ha.; 2502 cars in town.
C. Area is 17 ha.; 14,212 cars in town.

15. Mrs Cobb makes and sells her own apple-cranberry juice. In pitcher A, she mixed 4 cranberry flavoured cubes and 3 apple flavoured cubes, with some water. In pitcher B, she used 3 cranberry and 2 apple flavoured cubes in the same amount of water. If you ask for a drink that has a stronger cranberry taste, from which pitcher should she pour your drink?

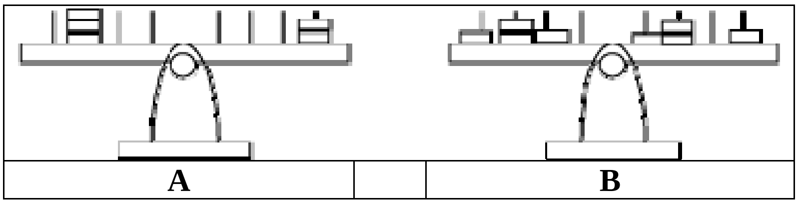
16. Jim's mother asked him to go to her desk and get his dad's picture and its enlargement, but when Jim went into her office, he found five pictures of his dad in various sizes:

A. 9cm x 10cm
B. 10cm x 12cm
C. 8cm x 9.6cm
D. 6cm x 8cm
E. 5cmx6.5cm

Which two did she want?

17. From Lewis Carroll: If 6 cats can kill 6 rats in 6 minutes, how many cats will be needed to kill 100 rats in 50 minutes?

18. Two identical balance beams are placed on a table and a number of weights are added while the beams are held in place. Would you expect each beam to tip toward the right or toward the left when it is released?



19. What is the ratio of men to women in a town where $^{2}/_{3}$ of the men are married to ¾ of the women?

20. In a gourmet coffee shop, two types of beans are combined and sold as the House Blend. One bean sells for $8.00 per kg and the other for $14.00 per kg. They mix up batches of 50 kg at a time and sell the House Blend for $10.00 a kg. How many kgs of each coffee go into the blend?

Multiplicative Thinking in Mathematics

- Ability to see situations in a multiplicative sense rather than an additive sense
- Flexibility in thinking about numbers and situations involving number

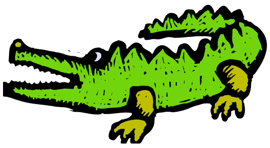
Additive Thinking - Describing a change from 2 to 10 as an addition of 8 *Multiplicative Thinking* – Describing a change from 2 to 10 as multiplying by 5

One of the key aspects of proportional thinking is being able to consider situations of change in both additive and multiplicative terms, adjusting appropriately according to the context. Being able to make comparisons in additive and multiplicative terms is also referred to as absolute and relative thinking respectively.

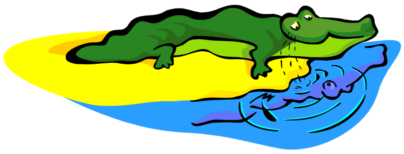
Consider the following situation:

> *At the zoo, there are two long-term resident crocodiles that have affectionately been named Prickles and Tiny. When they arrived at the zoo, Prickles was 4 metres long and Tiny was 5 metres long. Five years later, both crocodiles are now fully grown. Prickles is 7 metres long and Tiny is 8 metres long.*

Arriving at the zoo:

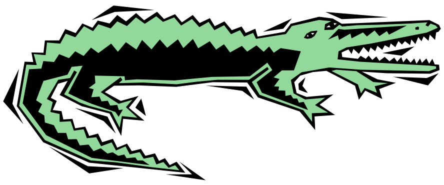


Prickles (4 metres)

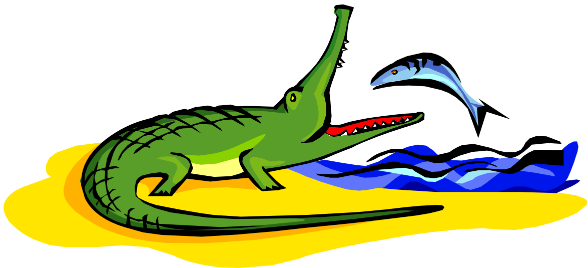


Tiny (5 metres)

Five years later, fully grown:



Prickles (7 metres)



Tiny (8 metres)

Have Prickles and Tiny grown by the same amount? In actual growth and independent of anything else, the two crocodiles have grown by the same amount. From an additive (absolute) perspective, both crocodiles have grown by the same amount of 3 metres. Another perspective is to consider the growth in relation to their beginning length. Prickles has grown 3 metres, or ¾ of his beginning length. Tiny has grown 3 metres, or $^3/_5$ of his beginning length. Prickles has grown more than Tiny because ¾ is > $^3/_5$. From a relative perspective, the crocodiles have not grown the same amount.

What if Prickles and Tiny had grown the same amount in relative terms? Tiny would have grown ¾ of his beginning length, or 3¾ metres more than his beginning length. Therfore, Tiny's length would have been 8¾ metres. To determine this length, multiplication was required, not addition.

The situation with Prickles and Tiny underlies one of the most basic principles required of students in order for them to reason proportionally. They must be able to describe change situations in both absolute and multiplicative terms.

Sometimes situations that appear to be proportional and therefore multiplicative, are actually additive. Consider the following:

> *Two cyclists, Harry and Freda, are cycling equally fast around the cycling track. Freda commenced cycling before Harry arrived at the track and had completed 9 laps when Harry had completed 3. When Freda has completed 15 laps, how many laps will Harry have completed?*

When given a similar problem, many adults mistakenly identified this situation as a proportion situation, and calculated the solution to be 45, when it should of course be 9. This is an indication of how often people become accustomed to 'school problems' and automatically apply learned procedures, sometimes inappropriately. When promoting multiplicative thinking, it is important to ensure that students' early additive thinking is not totally ignored. Spending time allowing students to discuss proportion situations in both additive and multiplicative terms, encourages mathematical thinking.

Can you think of a time when you were half your mother's age? Can you think of another time when you were half your mother's age? Why not?

Additive to multiplicative thinking
To promote multiplicative thinking, students should be provided with opportunities to discuss change situations in both absolute (additive) and relative (multiplicative) terms. Students naturally tend to consider change situations in additive terms, but when prompted through specific questions, they can come to discuss situations in both additive and multiplicative terms. Consider the following situation:

Examples of questions requiring additive thinking:

Who has more tazos, Ruby or Trent?
How many more tazos does Trent have than Ruby?
How many fewer tazos does Ruby have than Trent?

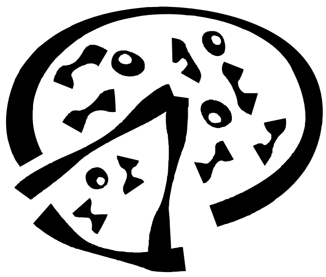
Examples of questions requiring multiplicative thinking:

How many times would you have to stack Ruby's tazos to get a pile as high as Trent's? [the answer is messy—did you mean it to be?]
What part of a dozen tazos does Ruby have?
What part of a dozen tazos does Trent have?
Both children have 3 super tazos. What percent of each child's tazos are super tazos?

Many fraction situations also lend themselves to discussions in terms of additive and multiplicative relationships:

A pizza is cut into 8 equal slices. Three people have two slices each

How many slices did they eat altogether?
How much (what fraction) of the pizza did they eat?



Sally works 3 days a week.

How many days does she work?
How much (what fraction) of a week does she not work?

Even though many children in the early primary years tend to only think of situations in additive terms, many children in the early childhood years can intuitively reason proportionally. This type of thinking is evident when they explore and play. For example, pouring and counting the number of cups of water to fill a bucket at bath time; putting equal numbers of sweets into party bags; comparing the number of coins and number of sweets that can be purchased, are early childhood experiences through which proportional reasoning is developed. Often before students begin formal schooling, they have intuitive multiplication knowledge, and hence have the capacity for proportional reasoning. Being able to state that 3 cups of water fills one bucket, then 9 cups would fill 3; or knowing that 2 candy sticks cost

5c, so you could buy 8 candy sticks for 20c, are both examples of proportional reasoning that many young children demonstrate when we take the time to listen to them.

The importance of proportional reasoning in mathematics

Proportional reasoning is fundamental to working with rational number concepts:

- Fractions
- Ratio
- Percentages
- Decimals
- Proportion

Multiplicative thinking provides access for investigating a range of problems:

Butterflies
At the zoo in the butterfly enclosure, five drops of nectar is sufficient for 2 butterflies. How many butterflies could be fed with 12 drops of nectar?

Retail Therapy
Oscar bought a shirt in an end-of–season sale for $79.95. The original price was covered by a 30% off sticker but the sign on top of the rack said "Now an additional 15% off already reduced prices." How could he work out how much he had saved? What percentage of the original cost did the end up paying?

Dingos
35 dingos were found in a 146 hectare section of farmland. 27 dingos were found in a 103 hectare section of farmland. Which section of farmland had the highest dingo population?

Enlargements
To fit a particular display space, a client wanted the width of an A4 image enlarged in the ratio 1:2.6 and the height enlarged in the ratio of 1:3.7. The photographic shop based their charges on area. How much was the client asked to pay for the enlarged image if the cost of an A4 image was $12.45?

Additive and Multiplicative Thinking
Queensland 1-10 Mathematics Syllabus

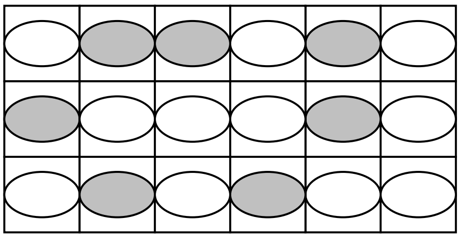
The development of multiplicative thinking is located within the Number Strand of the new 1-10 Mathematics Syllabus, and specifically within the substrand of Multiplication and Division. The Multiplication and Division substrand has a number of subcategories, outlining the importance of developing understanding at a conceptual level, as well as linking knowledge within this substrand. The subcategories within the Multiplication and Division substrand are as follows:

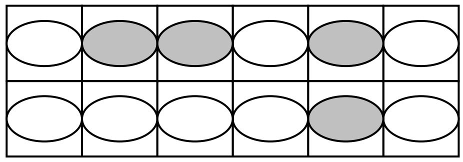
- Multiplication
- Division
- Connections
- Fractions & proportion
- Mental computation strategies
- Computational methods

Promoting discussion

The following are some situations that may be of value as discussion points for your class. As you read each one, think of the types of questions you could ask to promote analysis of each situation in additive and multiplicative terms.

1. Each of the cartons below contains some white eggs and some brown eggs. Which has more brown eggs?



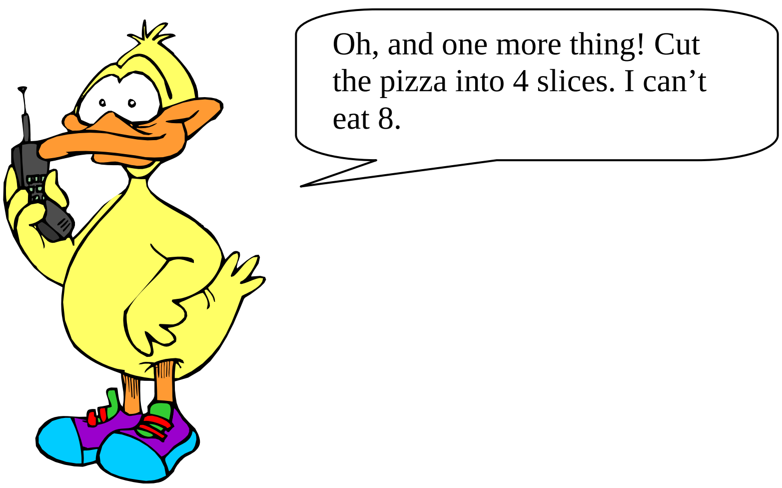


2. Dan and Tasha both started from home at the same time, 10am. Dan walked 2 km to the post office, 3 km from the post office to the zoo and then 1km from the zoo to home. Tasha walked 2.5 km to her friend's house, then 1.5 km from her friend's house to the supermarket, then 3 km from the supermarket to home. Both Dan and Tasha arrived home at exactly the same time, 12:30 pm. Describe exactly how you can tell who walked further, Dan or Tasha?

3. Here are some dimensions of three rectangles. Which one of them is most square?

Rectangle A: 75 cm x 114 cm
Rectangle B: 455 cm x 494 cm
Rectangle C: 284 cm x 245 cm

4. Analyse Fred's comment:



Analysing proportion situations

The fact that proportional reasoning permeates so many topics in mathematics has ensured that this has been an active field of mathematics educational research. Many researchers and educators have highlighted the similar features of proportion situations, and how this may serve as a way of assisting students to develop their proportional reasoning skills. One of the common features of proportion situations is that it is a comparison between two entities, or quantities. A label given to the quantities is that of 'measure spaces'. The term measure spaces basically refers to the two components of comparison in a proportion situation. Each measure space is labelled as M_1 and M_2 respectively. For example, when comparing the ratio of cordial to water in fruit juice mixes, or considering a rate such as the cost of petrol per litre, two components are being compared. When dealing with the quantities of the components, two types of analysis can occur: a "between" analysis and a "within" analysis. Consider the following example.

> A cordial drink is made up of a mixture of 1 part cordial to 5 parts water. How much water needs to be mixed with 40 mL of cordial to make a drink of the required strength?

The two measure-spaces (M_1 and M_2), are the cordial and the water respectively and represented as follows:

$\mathbf{M_1}$ cordial (mL)	$\mathbf{M_2}$ water (mL)
1	5
40	$\boldsymbol{x}$

Considering the situation from within the spaces, the new cordial amount is 40 times the original amount. To solve the proportion problem, use of the "within" strategy involves thinking that in the first space the quantity of cordial is multiplied by forty. Therefore, in the second measure space the quantity of water must also be multiplied by forty to maintain the relationship.

<u>Within</u>

	$\mathbf{M_1}$ cordial (ml)	$\mathbf{M_2}$ water (ml)	
	1	5	
× 40	↓	↓	× 40
	40	$\boldsymbol{x}$	

Considering the situation between the two measure-spaces, the amount of water (M_2) is five times that of the cordial (M_1). Therefore, to find how much water is needed when there is 40mL of cordial, this amount is multiplied by 5. The multiplicative basis of proportional thinking in the two measure spaces is apparent.

<u>Between</u>

$\mathbf{M_1}$ cordial (ml)		$\mathbf{M_2}$ water (ml)
1	→ × 5	5
40	→ × 5	$\boldsymbol{x}$

Rate situations can be thought of as structurally similar to the ratio situations, making use of the idea of proportion. For example, a speed of 60 km/h may be represented as in Figure 4.

	M_1	M_2	
	distance (km)	time (h)	
× 2 ↓	60	1	↓ × 2
	120	2	

Initially at least, a problem such as how far would one travel in 2½ hours at 80 km/h could be solved by thinking proportionally. The rate is a “between” measure-spaces “comparison”. A speed of 60 kilometres for every hour travelled is similar to 1 part of cordial for every 5 parts of water.

Proportional reasoning relies on multiplicative thinking, and fundamentally understanding of the operation of multiplication. Some advocates suggest that as proportional thinking is linked to multiplication, all multiplicative situations should be represented in symbolic form in a similar way. Representing multiplication situations using the measure spaces representation enables students to see connections between proportional situations (rate, ratio, percentages), and to develop computational strategies for solving such problems. In this way, multiplication and division is connected in a more systematic way. For example, the following rate problem can easily be represented using the measure spaces representation, and application of division supports successful attainment of the solution:

> A bulk pack of sweets contains 1.5 kg and costs $3.60. If the sweets are repackaged into 250 g packs, what price should each pack be to break even?

M_1	M_2
mass (g)	price ($)
1500(g)	$3.60
250(g)	x

Using a within strategy, it can be seen that the first quantity is divided by 6. Therefore, the cost for 250 g will be $3.60 ÷ 6 = $0.60.

When using the measure space representation for proportion problems, typically three elements are given and the task is to find the fourth element. Such proportional problems are generally referred to as “missing value” problems. Students in the middle years have often been taught to solve such problems by writing an algebraic equation to represent the problem and then solving the equation. The standard solution procedure for solving proportion equations is via algebraic means: “cross-multiply and solve for *x*.” The teaching of the standard algorithm however, is consistently criticised by many mathematics education researchers. The main criticism in “equationising” proportion situations is that the focus moves to equation solving rather than thinking in terms of the proportional nature of the problem situation. The proportion equation also uses fraction notation, although the proportion situation is rarely a fraction in the sense of a part to whole multiplicative comparison.

The multi-faceted nature of proportional reasoning

Proportional reasoning is not merely exemplified through flawless execution of procedures for solution to proportional situations. Proportional reasoning is multi-faceted. Proportional reasoning is being able to explain, to interpret, to recognise proportionality. It is also about representing proportional relationships in a number of ways, including in graphs, in tables, in equations. There are three main types of proportional relationships: simple direct proportion, indirect (inverse) proportion, and complex proportion.

Simple direct proportion
A direct proportion occurs when two quantities change in the same direction, that is, as one quantity increases (or decreases), the other quantity also increases (or decreases), but the relative size of the change stays the same. For example:

The ideal cordial mix is 2 cups of water for every $^{1}/_{3}$ cup of syrup. If you start with 8 cups of water, you would need $1^{1}/_{3}$ cups of syrup. The amount of water as compared to the amount of sugar is always the same no matter how big the batch of cordial made. There is always 6 times the amount of water compared to the amount of syrup. Looking at the situation from the other direction, there is always $^{1}/_{6}$ the amount of syrup for every cup of water.

Another example of a simple direct proportion is when you have 2 cups of coconut for every 3 cups of flour in a biscuit mixture. The amount of flour is always 1 ½ times the amount of coconut; the amount of coconut is always $^{2}/_{3}$ the amount of flour.

Inverse proportion
Inverse proportion is where one quantity increases relative to the way in which the other quantity decreases. This is often demonstrated in work situations where an increase in the number of people working on a job means that the time taken to complete the job will decrease (assuming that all workers put in a similar amount of effort). For example:

Suppose it takes 6 people 4 days to complete a job. If we double the number of people working, then the job should be completed in half the time. Another way of looking at this situation is to look at the product: as one factor increases (or decreases), the other factor decreases (or increases), but the product remains the same:

#people	#days	#person-days
6	4	=24
4	6	=24
12	2	=24
8	3	=24

Complex proportion
When proportions involve more than two variables, and one of the variables may change simultaneously in the same direction as one of the others and in the opposite direction from another we have a complex proportion situation. Consider the relationship between mass of an object and its volume. A small object may be quite heavy, or quite light (think of a cube of lead compared to a cube the same size, of wood). One of the cubes will have a greater density than the other. The relationship between the volume of an object (the amount of space it takes up) and its mass (how heavy it is) is the concept of density. There are two variables to consider in order to discuss the density of objects. Such understanding underpins floating and sinking – a core concept in science. Yet, if students have difficulty understanding simple proportion, they may struggle considerably when more than two variables must be considered simultaneously.

Ratio tables

One way of assisting students to develop mental strategies for solving proportion problems is through the use of ratio tables. Ratio tables are a convenient way of symbolising the elements within proportion situations, and for supporting thinking strategies for solution. Ratio tables encourage the use of number strategies such as halving, doubling, multiplying by 10, and so on. Some examples of ratio tables and how they can be used for solving proportion situations are below:

Example 1

There is 1 tent per 12 children. How many children can fit in 14 tents?

Tents	1	10	5	15	14
Children	12	120	60	180	168

Strategy: times by 10, divide by 2, multiply by 3, subtract 1 group of children.

Example 2

Bottles of water are packaged into 15 bottle boxes. How many bottles of water would there be in 16 packages?

Package	1	10	5	15	16
Bottles	15	150	75	225	240

Strategy: times by 10, divide by 2, multiply by 3, add 1 group of bottles.

Some students' solution strategies

Seedling plants come in boxes of 35 plants. How many plants would be in 16 boxes?

Alex's solution strategy

Boxes	1	2	4	8	16
Plants	35	70	140	280	560

Sasha's solution strategy

Boxes	1	10	2	6	16
Plants	35	350	70	210	560

Try some for yourself

Juice is packaged in cases of 14 bottles. How many bottles of juice are in 9 cases?

At school, 98 bottles of juice were required. How many cases needed to be bought?

Mangoes are 2 for $3. How many could you buy for $7.50?

Apples are 3 for $2. How much would 33 apples cost?

Summary

Many mathematical tasks and activities require proportional reasoning. Drawing a plan view of a house, a "mud map" of the path from home to school, and a plan of the school yard; sharing four pizzas among three people or two chocolate bars between three people; determining the better buy when 1 kg costs $3.50 and 1.5 kg costs $4.20; determining whether there is more chance of selecting red from a collection of 3 red and 4 blue compared to a collection of 6 red and 8 blue, all require proportional reasoning. The development of proportional reasoning is a gradual process, underpinned by increasingly more sophisticated multiplicative thinking and the ability to compare two quantities in relative (multiplicative) rather than absolute (additive) terms. Proportional reasoning as part of the multiplicative field has been identified as a key concept underlying a wide range of topics studied at the middle-school level. The task for middle-years teachers is to assist students to build, consolidate and link their proportional reasoning ability; not an easy task, as research consistently indicates students' difficulty with proportion related topics. Our goal in this project is explore and develop ways to support students' proportional thinking, and to assist them to make connections between core ideas within mathematics and science.

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Some material contained in these notes has been taken and adapted from Lamon (1999) (see reference list for complete citation). This includes:

- The 10 questions in the "Thinking about proportion" section
- The 10 questions included in the "Further thinking about proportional reasoning"
- The description of absolute and relative thinking with respect to the crocodiles
- The example in the "Additive to multiplicative thinking" about Ruby and Trent and their tazos, the pizza, and the description of Sally's week
- The four items for promoting discussion on additive and multiplicative thinking (eggs, Dan & Tasha, the rectangle, Fred's comment)
- Notes on inverse proportion

The cycling track situation is an adaptation of the oft-cited running track problem presented in Cramer, Post and Currier (1988).

The ratio tables examples have been taken from the National Science Foundation (1998) materials.

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ANSWERS

1. Six workers can build a house in 3 days. Assuming that all of the workers work at the same rate, how many workers would it take to build a house in 1 day? (18)
2. Eighty M&Ms will be divided between two boys in the ratio 2:3. How many M&Ms will child boy receive? (32:48)
3. If 5 chocolates cost $.75, how much do 13 cost? ($1.95)
4. Between them, John and Mark have 32 marbles. John has 3 times as many as Mark. How many marbles does each boy have? (John has 24, Mark has 8)
5. Jane loves to read. She can read a chapter in about 30 minutes. Assuming chapters are all about the same length, how long will it take her to read a book with 14 chapters? (7 hours)
6. Six students were given 20 minutes to clean up the classroom after an eraser fight. They were angry and named 3 other accomplices. The principal added their friends to the clean-up crew and changed the time limit. How much time did she give them to complete the job? (13 $^{1}/_{3}$ minutes)
7. If 1 football player weighs 115 kgs, what is the total weight of 11 starters? (assuming same mass - 1265 kgs)
8. Sandra wants to buy an MP3 player costing $210. Her mother agreed to pay $5 for every $2 Sandra saved. How much did each pay? ($150; $60)
9. A company usually sends 9 people to install a security system in an office building, and they do it in about 96 minutes. Today, they have only three people to do the same size job. How much time should be scheduled to complete the job? (288 minutes)
10. A motor bike can run for 10 minutes on $.30 worth of fuel. How long could it run on $1.05 worth of fuel? (38$^{1}/_{3}$ minutes)
11. On a sunny day, you and your friend were taking a long walk. You got tired and stopped near a telephone pole for a little rest, but your nervous friend couldn't stand still. He pace out your shadow and discovered it was 5 metres long even though you are really only 1.5 metres tall. He paced the long shadow of the telephone pole and found that it was 35 metres long. He wondered how high the telephone pole really is. Can you figure it out? (10 $^{2}/_{3}$ m)
12. Which is more square, a rectangle that measures 35cm x 39cm or a rectangle that measures 22cm x 25cm? (35cm x 39cm)
13. Two gears, A and B, are arranged so that the teeth of one gear mesh with the teeth of another. Gear A turns clockwise and has 54 teeth. Gear B turns counter clockwise and has 36 teeth. If gear A makes 5.5 rotations, how many turns will gear B make? (8.25)
14. Mr Brown is a bike rider. He considered living in Allentown, Binghamton, and Chester. In the end, he chose Binghamton because, as he put it, "All else being equal, I chose the town where bikes stand the greatest chance on the roads against cars." Is Binghamton town A, B, or C?
    A. Area is 15 ha.; 12,560 cars in town.
    B. Area is 3 ha.; 2502 cars in town.
    C. Area is 17 ha.; 14,212 cars in town. (B)
15. Mrs Cobb makes and sells her own apple-cranberry juice. In pitcher A, she mixed 4 cranberry flavoured cubes and 3 apple flavoured cubes, with some water. In pitcher B, she used 3 cranberry and 2 apple flavoured cubes in the same amount of water. If you ask for a drink that has a stronger cranberry taste, from which pitcher should she pour your drink? (B)
16. Jim's mother asked him to go to her desk and get his dad's picture and its enlargement, but when Jim went into her office, he found five pictures of his dad in various sizes:
    A. 9cm x 10cm

   B. 10cm x 12cm
   C. 8cm x 9.6cm
   D. 6cm x 8cm
   E. 5cmx6.5cm

   Which two did she want? (B & C)

17. From Lewis Carroll: If 6 cats can kill 6 rats in 6 minutes, how many cats will be needed to kill 100 rats in 50 minutes? (12 cats)
18. Two identical balance beams are placed on a table and a number of weights are added while the beams are held in place. Would you expect each beam to tip toward the right or toward the left when it is released? (A to the left; B to the left)
19. What is the ratio of men to women in a town where $^2/_3$ of the men are married to ¾ of the women? (9 men:8 women)
20. In a gourmet coffee shop, two types of beans are combined and sold as the House Blend. One bean sells for $8.00 per kg and the other for $14.00 per kg. They mix up batches of 50 kg at a time and sell the House Blend for $10.00 a kg. How many kgs of each coffee go into the blend? (Coffee A 33 $^1/_3$ kg; Coffee B 16 $^2/_3$ kg)