MC SAM
Making connections: Science and Mathematics

An integrated Science and Mathematics exploration
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Making Connections: Science and Mathematics
The MC SAM PROJECT

The MC SAM project was a partnership research project with School of Education lecturers from The University of Queensland and teachers from six schools in the Brisbane region (2007-2009). It was jointly funded by the Australian Research Council (ARC), The University of Queensland and the School Partners. The official title of the project was:

Learning essential knowledge by design: Promoting and connecting mathematics and science in the middle years of schooling.

The working title of the project throughout its implementation was The MC SAM Project (Making Connections – Science and Mathematics). The project brought together teachers from primary and secondary schools who taught mathematics and science. The focus was on the development of classroom learning activities to promote students’ proportional reasoning as required in science and mathematics topics.

The three year project, an ARC Linkage grant, involved the following six state and private schools in Queensland:

- All Hallows School Brisbane,
- Bundamba State Secondary College,
- Bremer State High School,
- Faith Lutheran College Redlands,
- Kenmore State High School, and
- St Peter’s Catholic Primary School Rochedale.

The research team included:

- Dr Shelley Dole and Dr Tony Wright from The University of Queensland,
- Professor Doug Clarke from Australian Catholic University (Melbourne),
- Mr Geoff Hilton, from The University of Queensland, was awarded an Australian Postgraduate Scholarship through this project and became an integral part of the research team whilst completing his doctorate.

Project Background

As key learning areas within the curriculum, quality mathematics and science education is vital for all students, with key ideas providing the basis of essential knowledge for future citizenship (Millar & Osborne, 1998). However, it is well documented that many students in the middle years encounter difficulties with the study of these two subjects, and this has led to repeated calls for new approaches to curriculum, pedagogy and assessment (e.g., Kilpatrick, Swafford & Findell, 2001; Committee on Science, Engineering and Public Policy, 2005). It is in the middle years of schooling that essential ideas must be made explicit in the mathematics and science curriculum, so that students exit compulsory schooling with rich, connected conceptualisations in these domains. The challenge to teachers is to connect these essential ideas in students who increasingly are living digital and global lifestyles.

This project aimed to explicate the common core themes of mathematics and science, with particular emphasis on proportional reasoning. Proportional reasoning is fundamental to both
mathematics and science (Lamon, 2005), underpinning many topics in the middle years curriculum (e.g., scale drawing, surface area/volume ratio, probability, molarity, force and motion, algebra, fractions). Proportional reasoning refers to the capacity to understand the relationships in proportional situations and to work meaningfully with them (Lesh, Post and Behr, 1988). For example, determining the best value out of 200 grams for $3 and 250 grams for $4; or understanding why a baby locked in a car on a hot day suffers more than an adult in the same circumstances, involve proportional reasoning. Despite its centrality to mathematics and science, research has continually revealed that many students in the middle years struggle with proportion-related topics (e.g., Behr, Harel, Post & Lesh, 1992; Ben-Chaim, Fey, Fitzgerald, Benedetto & Miller, 1998; Lo & Watanabe, 1997). Such underdeveloped proportional reasoning potentially impacts real-world situations, sometimes with life-threatening or disastrous consequences (e.g., administration of incorrect doses in medicine [Preston, 2004]; inaccurate mix of chemicals in pesticides; failure to accurately convert between units of metric and imperial units of measure).

Proportional reasoning clearly permeates both the mathematics and science curriculum, yet typically, these two subjects are taught separately in Australian schools and in the secondary school context in particular, often by two different subject specialist teachers. The Association for the Advancement of Science (AAAS)(2001), has stated that scientific literacy is promoted through linking mathematics, science and technology curriculum, and that curriculum should not comprise isolated bits of information, but as a “rich fabric of mutually supporting ideas and skills that develops over time” (p. 3). According to AAAS, two components underpin proportional reasoning: (1) ratios and proportionality (part/whole knowledge, relationships, computation) and (2) describing change (knowledge of related changes, kinds of change and invariance). Proportional reasoning also links to representing change graphically and symbolically. Describing change links to understanding topics associated with Atoms and Molecules, Laws of Motion, Systems, Natural Selection. With a more interconnected view of curriculum, teachers are in a much stronger position to create learning environments that assist students to link knowledge and develop rich conceptual bases of understanding. One of Bell’s (1993) principles of effective mathematics teaching is assisting students to make connections between isolated pieces of knowledge. This principle is equally valid for all curriculum areas and is one of the principles to be applied in this project. This project thus supported the advancement of curriculum reform at the grass-roots level. Its aim was to provided evidence-based research to validate components of proportional reasoning through teaching mathematics and science.

This project brought together teachers and educators to explore the synergies between mathematics and science curriculum through proportional reasoning; to create and implement innovative and engaging learning experiences and assessment strategies across mathematics and science topics, and to reflect upon their classroom practice and the learning outcomes of their students.

This booklet provides an overview of proportional reasoning and its importance in the mathematics and science curriculum, and includes some of the classroom activities used in this project to promote proportional reasoning. It also contains copies of a proportional reasoning assessment test that was developed as part of this project.
Proportional Reasoning

Introduction

Many topics within the school mathematics and science curriculum require knowledge and understanding of ratio and proportion. In mathematics, for example, problem solving and calculation activities in domains involving scale, probability, percent, rate, trigonometry, equivalence, measurement, the geometry of plane shapes, algebra are assisted through ratio and proportion knowledge. In science, calculations for density, molarity, speed and acceleration, force, require competence in ratio and proportion. As ratio and proportion permeate so many topics in mathematics and science, the importance of study of these two concepts in the school curriculum is highlighted.

In the middle years of schooling, ratio and proportion are typically studied in mathematics classes. In fact, ratio and proportion have been described as the cornerstone of middle years mathematics curriculum. However, research has consistently highlighted students’ difficulties with proportion and proportion-related tasks and applications, which means that many students will struggle with topics within both the middle years mathematics and science curriculum due to their lack of understanding of ratio and proportion. Understanding ratio and proportion is more than merely being able to perform appropriate calculations and being able to apply rules and formulae, and manipulating numbers and symbols in proportion equations. Educators are well aware that students’ computational performances are not a true indicator of the degree to which they understand the concepts underlying the calculations. Understanding in mathematics is generally described as a way of knowing. Knowing ratio and proportion is about proportional reasoning.

Proportional Reasoning

Before considering proportional reasoning, consider the meaning of the words ratio and proportion. In its barest form, ratio describes a situation in comparative terms, and proportion is when this comparison is used to describe a related situation in the same comparative terms. For example, if we say that the ratio of boys to girls in the class is 2 to 3, we are comparing the number of boys to the number of girls.

When we know that there are 30 children in the class we know that, proportionally, the number of boys is 12 and the number of girls is 18.

We are using the base comparison to apply it to the whole situation. In order to understand this relationship proportional reasoning is used. Proportional thinking and reasoning is knowing the multiplicative relationship between the base ratio and the proportional situation to which it is applied.

Proportional reasoning is being able to make comparisons between entities in multiplicative terms. This means that the relationship between the two entities is conceptualised as a multiplicative relationship.

For many young children, comparisons between entities are described in additive terms, and they compare groups using additive or subtractive language. For example, when comparing the number of boys to girls as in the example above (ratio of boys to girls 2:3), they may say that there is always one extra girl for each group of boys. So, if there were 4 boys, there would be five girls.
Being able to describe proportional situations using multiplicative language is an indicator of proportional reasoning.

The development of proportional reasoning is something that takes time. It is fostered by quality learning experiences in which students have opportunities to explore, discuss and experiment with proportion situations. Proportional reasoning is also dependent upon sound foundations of associated topics, particularly multiplication and division. Other mathematical topics through which proportional reasoning grows is through the study of rational number topics including fractions, decimals, percentages, scale drawing, and of course ratio and proportion. Proportional reasoning is fostered through rich conceptual understanding of ratio and proportion, but these are difficult concepts that present a challenge to both students and their teachers.

**Proportional Reasoning in Science and Maths**

Mathematics and science educators are increasingly talking about proportional reasoning as a fundamental link between mathematics and science. As outlined above, proportional reasoning is required to operate in many mathematics and science topics. Proportional reasoning is also fundamental to real-world and every-day situations, and hence underpins numeracy and scientific reasoning. For example, determining comparing 200 grams for $3 or 250 grams for $4; or understanding why a baby locked in a car on a hot day suffers more than an adult in the same circumstances, involves proportional reasoning. Drawing a plan view of a house, a “mud map” of the path from home to school, and a plan of the school yard; sharing four pizzas between three people or two chocolate bars between three people; determining the better buy when 1 kg costs $3.50 and 1.5 kg costs $4.20; determining whether there is more chance of selecting red from a collection of 3 red and 4 blue compared to a collection of 6 red and 8 blue, all require proportional reasoning. Underdeveloped proportional reasoning potentially impacts real-world situations, sometimes with life-threatening or disastrous consequences (e.g., administration of incorrect doses in medicine; inaccurate mix of chemicals in pesticides; failure to accurately convert between units of metric and imperial units of measure). A rich field of potential investigations that link science and mathematics opens up when one considers the multi-faceted and multi-dimensional nature of proportional reasoning.

In the middle years of schooling, instruction must take an active role in supporting the development of proportional reasoning. It is estimated that only 50% of the adult population can reason proportionally. The question must be asked as to why this is the case. As proportional reasoning permeates so many topics in science and mathematics, it is timely to consider our role as teachers in fostering students’ proportional reasoning skills. Proportional reasoning does not develop in a linear fashion and is something that is never an absolute state. It will continue to grow and develop with experience and explicit teaching. Mathematics and science teachers working together can mutually support each other in developing rich learning experiences that assist students to make connections between key topics across these two disciplines as well as foster numeracy and scientific thinking and reasoning.

**Thinking about Proportional Reasoning**

On the following pages are some proportional reasoning exercises taken from the following source: Lamon, S. (2005). *Teaching fractions and ratios for understanding*. (2nd ed.). Nahwah, NJ: Erlbaum. Try to do them in your head. As you are solving them, think of the strategies you are employing. These tasks all require proportional reasoning. Rather than
trying to remember school procedures or rules you may have learnt to do these tasks. Just try to jot down things to assist your memory without resorting to completion of pen and paper methods. Work through them on your own first before you share your thinking with others.

1. Six workers can build a house in 3 days. Assuming that all of the workers work at the same rate, how many workers would it take to build a house in 1 day?

2. Eighty M&Ms will be divided between two children in the ratio 2:3. How many M&Ms will each child receive?

3. If 5 chocolates cost $.75, how much do 13 cost?

4. Between them, John and Mark have 32 marbles. John has 3 times as many as Mark. How many marbles does each boy have?

5. Jane loves to read. She can read a chapter in about 30 minutes. Assuming chapters are all about the same length, how long will it take her to read a book with 14 chapters?

6. Six students were given 20 minutes to clean up the classroom after an eraser fight. They were angry and named 3 other accomplices. The principal added their friends to the clean-up crew and changed the time limit. How much time did she give them to complete the job?

7. If 1 football player weighs 115 kg, what is the total weight of 11 starters?

8. Sandra wants to buy an MP3 player costing $210. Her mother agreed to pay $5 for every $2 Sandra saved. How much did each pay?

9. A company usually sends 9 people to install a security system in an office building, and they do it in about 96 minutes. Today, they have only three people to do the same size job. How much time should be scheduled to complete the job?

10. A motor bike can run for 10 minutes on $.30 worth of fuel. How long could it run on $1.05 worth of fuel?

More proportional reasoning tasks

These problems are a bit more challenging. They have also been taken from Lamon (2005). Take your time on these problems. Pick and choose ones that appeal to you. The answers are contained on the following pages.

11. On a sunny day, you and your friend were taking a long walk. You got tired and stopped near a telephone pole for a little rest, but your nervous friend couldn’t stand still. He pace out your shadow and discovered it was 5 metres long even though you are really only 1.5 metres tall. He paced the long shadow of the telephone pole and found that it was 35 metres long. He wondered how high the telephone pole really is. Can you figure it out?

12. Which is more square, a rectangle that measures 35 cm x 39 cm or a rectangle that measures 22 cm x 25 cm?

13. Two gears, A and B, are arranged so that the teeth of one gear mesh with the teeth of another. Gear A turns clockwise and has 54 teeth. Gear B turns counter clockwise and has 36 teeth. If gear A makes 5.5 rotations, how many turns will gear B make?
14. Mr Brown is a bike rider. He considered living in Allentown, Binghamton, and Chester. In the end, he chose Binghamton because, as he put it, “All else being equal, I chose the town where bikes stand the greatest chance on the roads against cars.” Is Binghamton town A, B, or C?

A. Area is 15 ha.; 12 560 cars in town.
B. Area is 3 ha.; 2 502 cars in town.
C. Area is 17 ha.; 14 212 cars in town

15. Mrs Cobb makes and sells her own apple-cranberry juice. In pitcher A, she mixed 4 cranberry flavoured cubes and 3 apple flavoured cubes, with some water. In pitcher B, she used 3 cranberry and 2 apple flavoured cubes in the same amount of water. If you ask for a drink that has a stronger cranberry taste, from which pitcher should she pour your drink?

16. Jim’s mother asked him to go to her desk and get his dad’s picture and its enlargement, but when Jim went into her office, he found five pictures of his dad in various sizes:

A. 9 cm x 10 cm
B. 10 cm x 12 cm
C. 8 cm x 9.6 cm
D. 6 cm x 8 cm
E. 5 cm x 6.5 cm

Which two did she want?

17. From Lewis Carroll: If 6 cats can kill 6 rats in 6 minutes, how many cats will be needed to kill 100 rats in 50 minutes?

18. Two identical balance beams are placed on a table and a number of weights are added while the beams are held in place. Would you expect each beam to tip toward the right or toward the left when it is released?

19. What is the ratio of men to women in a town where \( \frac{2}{3} \) of the men are married to \( \frac{3}{4} \) of the women?

20. In a gourmet coffee shop, two types of beans are combined and sold as the House Blend. One bean sells for $8.00 per kg and the other for $14.00 per kg. They mix up batches of 50 kg at a time and sell the House Blend for $10.00 a kg. How many kgs of each coffee go into the blend?
Answers
1. 18
2. 32 and 48
3. $1.95
4. John has 24, Mark has 8.
5. 7 hours
6. 15 minutes
7. 1265 kg
8. Sandra paid $60, her mother paid $150.
9. 288 minutes
10. 35 minutes
11. 10.5 metres
12. The rectangle that measures 35 cm x 39 cm is more square.
13. 8.25
14. B has the lowest density of cars.
15. Pitcher B has the strongest flavour.
16. Jim’s mother wanted the original picture, C, and it’s enlargement B.
17. 200 cats
18. Both tip to the left.
19. The ratio is 9 men to 8 women.
20. Blend A 33 \( \frac{1}{3} \) kg; Blend B 16 \( \frac{2}{3} \) kg
**Multiplicative Thinking in Mathematics**

- Ability to see situations in a multiplicative sense rather than an additive sense
- Flexibility in thinking about numbers and situations involving number

One of the key aspects of proportional thinking is being able to consider situations of change in both additive and multiplicative terms, adjusting appropriately according to the context. Being able to make comparisons in additive and multiplicative terms is also referred to as absolute and relative thinking respectively.

<table>
<thead>
<tr>
<th>Additive Thinking</th>
<th>Multiplicative Thinking</th>
</tr>
</thead>
<tbody>
<tr>
<td>Describing a change from 2 to 10 as an addition of 8</td>
<td>Describing a change from 2 to 10 as multiplying by 5</td>
</tr>
</tbody>
</table>

Absolute and relative thinking has been exemplified by Lamon (2005) in a discussion about two snakes. Here, we have adapted Lamon's discussion to two crocodiles.

*At the zoo, there are two long-term resident crocodiles that have affectionately been named Prickles and Tiny. When they arrived at the zoo, Prickles was 4 metres long and Tiny was 5 metres long. Five years later, both crocodiles are now fully grown. Prickles is 7 metres long and Tiny is 8 metres long.*

<table>
<thead>
<tr>
<th>Arrival</th>
<th>5 years later</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1" alt="Image of Arrival" /></td>
<td><img src="image2" alt="Image of 5 years later" /></td>
</tr>
</tbody>
</table>

Have Prickles and Tiny grown by the same amount? In actual growth, the two crocodiles have grown by the same amount. From an additive (absolute) perspective, both crocodiles have grown by the same amount of 3 metres.

<table>
<thead>
<tr>
<th>Tiny at arrival</th>
<th>Tiny's growth</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image3" alt="Image of Tiny at arrival" /></td>
<td><img src="image4" alt="Image of Tiny's growth" /></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Prickles at arrival</th>
<th>Prickles' growth</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image5" alt="Image of Prickles at arrival" /></td>
<td><img src="image6" alt="Image of Prickles' growth" /></td>
</tr>
</tbody>
</table>
Another perspective is to consider the growth in relation to their beginning length. Prickles has grown 3 metres, or \( \frac{3}{4} \) of his beginning length. Tiny has grown 3 metres, or \( \frac{3}{5} \) of his beginning length. Prickles has grown more than Tiny because \( \frac{3}{4} \) is greater than \( \frac{3}{5} \). From a relative perspective, the crocodiles have not grown the same amount.

The situation with Prickles and Tiny requires a dynamic view of a situation, where the current state is considered in relation to the beginning state. This is a situation of change, and describing change situations in both absolute (additive) and relative (multiplicative) terms is an indicator of proportional reasoning.

Sometimes situations that appear to be proportional and therefore multiplicative, are actually additive. Consider the following (cited by Cramer, Post and Currier, 1988):

Two cyclists, Harry and Freda, are cycling equally fast around the cycling track. Freda commenced cycling before Harry arrived at the track and had completed 9 laps when Harry had completed 3. When Freda has completed 15 laps, how many laps will Harry have completed?

Cramer, Post and Currier (1988) found that, when given this problem, many adults mistakenly identify this situation as a proportion situation, and calculate the solution to be 45, when it should of course be 9. This is an indication of how often people become accustomed to ‘school problems’ and automatically apply learned procedures, sometimes inappropriately. When promoting multiplicative thinking, it is important to ensure that students’ early additive thinking is not totally ignored. Spending time allowing students to discuss proportion situations in both additive and multiplicative terms, encourages mathematical thinking.

Can you think of a time when you were half your mother’s age? Can you think of another time when you were half your mother’s age? Why not? Take some time to think about this. The only time you will be/were half your mother’s age is when the same number of years have passed as the age your mother was when you were born. For example, if your mother was 25 when you were born, in 25 years, when your mother is 50, you will be half your mother’s age. When your mother is 75, you will be 25 years younger (50). You won’t be half your mother’s age again as this is an additive relationship.

**Additive to multiplicative thinking**

To promote multiplicative thinking, students should be provided with opportunities to discuss change situations in both absolute (additive) and relative (multiplicative) terms. Students naturally tend to consider change situations in additive terms, but when prompted through specific questions, they can come to discuss situations in both additive and multiplicative terms. The following examples have been taken from Lamon (2005). The picture below shows two children comparing the number of tazos they each have.
Examples of questions requiring additive thinking:
- Who has more tazos, Ruby or Trent?
- How many more tazos does Trent have than Ruby?
- How many fewer tazos does Ruby have than Trent?

Examples of questions requiring multiplicative thinking:
- How many times would you have to stack Ruby’s tazos to get a pile as high as Trent’s?
- What part of a dozen tazos does Ruby have?
- What part of a dozen tazos does Trent have?
- Both children have 3 super tazos in their collection. What fraction of each child’s tazos are super tazos?

Many fraction situations also lend themselves to discussions in terms of additive and multiplicative relationships:

A pizza is cut into 8 equal slices. Three people have two slices each
- How many slices did they eat altogether?
- How much (what fraction) of the pizza did they eat?

Sally works 3 days a week.
- How many days does she work?
- How much (what fraction) of a week does she not work?

Even though many children in the early primary years tend to only think of situations in additive terms, many children in the early childhood years can intuitively reason proportionally. This type of thinking is evident when they explore and play. For example, pouring and counting the number of cups of water to fill a bucket at bath time; putting equal numbers of sweets into party bags; comparing the number of coins and number of sweets that can be purchased, are early childhood experiences through which proportional reasoning is developed. Often before students begin formal schooling, they have intuitive multiplication knowledge, and hence have the capacity for proportional reasoning. Being able to state that 3 cups of water fills one bucket, then 9 cups would fill 3; or knowing that 2 candy sticks cost 5c, so you could buy 8 candy sticks for 20c, are both examples of proportional reasoning that many young children demonstrate when we take the time to listen to them.


Promoting discussion

The following are some situations that may be of value as discussion points for your class. They have also been taken from Lamon (2005). As you read each one, think of the types of questions you could ask to promote analysis of each situation in additive and multiplicative terms.

1. Each of the cartons below contains some white eggs and some brown eggs. Which has more brown eggs?

![Cartons of eggs]

2. Dan and Tasha both started from home at the same time, 10am. Dan walked 2 km to the post office, 3 km from the post office to the zoo and then 1km from the zoo to home. Tasha walked 2.5 km to her friend’s house, then 1.5 km from her friend’s house to the supermarket, then 3 km from the supermarket to home. Both Dan and Tasha arrived home at exactly the same time, 12:30 pm. Describe exactly how you can tell who walked further, Dan or Tasha?

3. Here are some dimensions of three rectangles. Which one of them is most square?

- Rectangle A: 75 cm x 114 cm
- Rectangle B: 455 cm x 494 cm
- Rectangle C: 284 cm x 245 cm

4. Analyse Fred’s comment:

Oh, and one more thing! Cut the pizza into 4 slices. I can’t eat it.
**Analysing proportion situations**

The fact that proportional reasoning permeates so many topics in mathematics has ensured that this has been an active field of mathematics educational research. Many researchers and educators have highlighted the similar features of proportion situations, and how this may serve as a way of assisting students to develop their proportional reasoning skills. The following overview is elaborated in Shield, M. & Dole, S. (2008). Proportion in middle-school mathematics – It’s everywhere. *Australian Mathematics Teacher, 64*(3), 10-15. One of the common features of proportion situations is that it is a comparison between two entities, or quantities. A label given to the quantities is that of ‘measure spaces’. The term measure spaces basically refers to the two components of comparison in a proportion situation. Each measure space is labelled as M_1 and M_2 respectively. For example, when comparing the ratio of cordial to water in fruit juice mixes, or considering a rate such as the cost of petrol per litre, two components are being compared. When dealing with the quantities of the components, two types of analysis can occur: a *between* analysis and a *within* analysis. Consider the following example.

A cordial drink is made up of a mixture of 1 part cordial to 5 parts water. How much water needs to be mixed with 40 mL of cordial to make a drink of the required strength?

The two measure-spaces (M_1 and M_2), are the cordial and the water respectively and represented as follows:

<table>
<thead>
<tr>
<th>M_1</th>
<th>M_2</th>
</tr>
</thead>
<tbody>
<tr>
<td>cordial (mL)</td>
<td>water (mL)</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>40</td>
<td>x</td>
</tr>
</tbody>
</table>

Considering the situation from *within* the spaces, the new cordial amount is 40 times the original amount. To solve the proportion problem, use of the *within* strategy involves thinking that in the first space the quantity of cordial is multiplied by forty. Therefore, in the second measure space the quantity of water must also be multiplied by forty to maintain the relationship.

<table>
<thead>
<tr>
<th>M_1</th>
<th>M_2</th>
</tr>
</thead>
<tbody>
<tr>
<td>cordial (mL)</td>
<td>water (mL)</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>40</td>
<td>x</td>
</tr>
</tbody>
</table>

Considering the situation *between* the two measure-spaces, the amount of water (M_2) is five times that of the cordial (M_1). Therefore, to find how much water is needed when there is 40mL of cordial, this amount is multiplied by 5. The multiplicative basis of proportional thinking in the two measure spaces is apparent.
Rate situations can be thought of as structurally similar to the ratio situations, making use of the idea of proportion. For example, a speed of 60 km/h may be represented below.

<table>
<thead>
<tr>
<th>$M_1$</th>
<th>$M_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>cordial (mL)</td>
<td>water (mL)</td>
</tr>
<tr>
<td>1</td>
<td>$\times 5$</td>
</tr>
<tr>
<td>40</td>
<td>$\times 5$</td>
</tr>
</tbody>
</table>

Initially at least, a problem such as how far would one travel in $2\frac{1}{2}$ hours at 80 km/h could be solved by thinking proportionally. The rate is a *between* measure-spaces *comparison*. A speed of 60 kilometres for every hour travelled is similar to 1 part of cordial for every 5 parts of water.

Proportional reasoning relies on multiplicative thinking, and fundamentally understanding of the operation of multiplication. Some advocates suggest that as proportional thinking is linked to multiplication, all multiplicative situations should be represented in symbolic form in a similar way. Representing multiplication situations using the measure spaces representation enables students to see connections between proportional situations (rate, ratio, percentages), and to develop computational strategies for solving such problems. In this way, multiplication and division is connected in a more systematic way. For example, the following rate problem can easily be represented using the measure spaces representation, and application of division supports successful attainment of the solution:

A bulk pack of sweets contains 1.5 kg and costs $3.60. If the sweets are repackaged into 250g packs, what price should each pack be to break even?

<table>
<thead>
<tr>
<th>$M_1$</th>
<th>$M_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>mass (g)</td>
<td>price ($)</td>
</tr>
<tr>
<td>1500</td>
<td>$3.60$</td>
</tr>
<tr>
<td>250</td>
<td>$x$</td>
</tr>
</tbody>
</table>

Using a *within* strategy, it can be seen that the first quantity is divided by 6. Therefore, the cost for 250 g will be $3.60 \div 6 = \$0.60$. 

13
When using the measure space representation for proportion problems, typically three elements are given and the task is to find the fourth element. Such proportional problems are generally referred to as missing value problems. Students in the middle years have often been taught to solve such problems by writing an algebraic equation to represent the problem and then solving the equation. The standard solution procedure for solving proportion equations is via algebraic means: “cross-multiply and solve for x.” The teaching of the standard algorithm however, is consistently criticised by many mathematics education researchers. The main criticism in equationising proportion situations is that the focus moves to equation solving rather than thinking in terms of the proportional nature of the problem situation. The proportion equation also uses fraction notation, although the proportion situation is rarely a fraction in the sense of a part to whole multiplicative comparison.

**The multi-faceted nature of proportional reasoning**

Proportional reasoning is not merely exemplified through flawless execution of procedures for solution to proportional situations. Proportional reasoning is multi-faceted. Proportional reasoning is being able to explain, to interpret, to recognise proportionality. It is also about representing proportional relationships in a number of ways, including in graphs, in tables, in equations. There are three main types of proportional relationships: simple direct proportion, indirect (inverse) proportion, and complex proportion.

**Simple direct proportion**

A direct proportion occurs when two quantities change in the same direction, that is, as one quantity increases (or decreases), the other quantity also increases (or decreases), but the relative size of the change stays the same. For example:

The ideal cordial mix is 2 cups of water for every \( \frac{1}{3} \) cup of syrup. If you start with 8 cups of water, you would need \( 1 \frac{1}{3} \) cups of syrup. The amount of water as compared to the amount of sugar is always the same no matter how big the batch of cordial made. There is always 6 times the amount of water compared to the amount of syrup. Looking at the situation from the other direction, there is always \( \frac{1}{6} \) the amount of syrup for every cup of water.

Another example of a simple direct proportion is when you have 2 cups of coconut for every 3 cups of flour in a biscuit mixture. The amount of flour is always 1 ½ times the amount of coconut; the amount of coconut is always \( \frac{2}{3} \) the amount of flour.

**Inverse proportion**

Inverse proportion is where one quantity increases relative to the way in which the other quantity decreases. This is often demonstrated in work situations where an increase in the number of people working on a job means that the time taken to complete the job will decrease (assuming that all workers put in a similar amount of effort). For example:

Suppose it takes 6 people 4 days to complete a job. If we double the number of people working, then the job should be completed in half the time. Another way of looking at this situation is to look at the product: as one factor increases (or decreases), the other factor decreases (or increases), but the product remains the same:
<table>
<thead>
<tr>
<th>people</th>
<th>days</th>
<th>person-days</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>4</td>
<td>= 24</td>
</tr>
<tr>
<td>4</td>
<td>6</td>
<td>= 24</td>
</tr>
<tr>
<td>12</td>
<td>2</td>
<td>= 24</td>
</tr>
<tr>
<td>8</td>
<td>3</td>
<td>= 24</td>
</tr>
</tbody>
</table>

**Complex proportion**

When proportions involve more than two variables, and one of the variables may change simultaneously in the same direction as one of the others and in the opposite direction from another we have a complex proportion situation. Consider the relationship between mass of an object and its volume. A small object may be quite heavy, or quite light (think of a cube of lead compared to a cube the same size, of wood). One of the cubes will have a greater density than the other. The relationship between the volume of an object (the amount of space it takes up) and its mass (how heavy it is) is the concept of density. There are two variables to consider in order to discuss the density of objects. Such understanding underpins floating and sinking — a core concept in science. Yet, if students have difficulty understanding simple proportion, they may struggle considerably when more than two variables must be considered simultaneously.

**Ratio tables**

One way of assisting students to develop mental strategies for solving proportion problems is through the use of ratio tables. Ratio tables are a convenient way of symbolising the elements within proportion situations, and for supporting thinking strategies for solution. Ratio tables encourage the use of number strategies such as halving, doubling, multiplying by 10, and so on. Some examples of ratio tables and how they can be used for solving proportion situations are below:

**Example 1**

There is 1 tent per 12 children. How many children can fit in 14 tents?

<table>
<thead>
<tr>
<th>Tents</th>
<th>1</th>
<th>10</th>
<th>5</th>
<th>15</th>
<th>14</th>
</tr>
</thead>
<tbody>
<tr>
<td>Children</td>
<td>12</td>
<td>120</td>
<td>60</td>
<td>180</td>
<td>168</td>
</tr>
</tbody>
</table>

Strategy: times by 10, divide by 2, multiply by 3, subtract 1 group of children.

**Example 2**

Bottles of water are packaged into 15 bottle boxes. How many bottles of water would there be in 16 packages?

<table>
<thead>
<tr>
<th>Package</th>
<th>1</th>
<th>10</th>
<th>5</th>
<th>15</th>
<th>16</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bottles</td>
<td>15</td>
<td>150</td>
<td>75</td>
<td>225</td>
<td>240</td>
</tr>
</tbody>
</table>

Strategy: times by 10, divide by 2, multiply by 3, add 1 group of bottles.
Add notes to the arrows to indicate the strategy

Some students’ solution strategies.
The following two ration tables show how Alex and Sasha each solve the problem.

Seedling plants come in boxes of 35 plants. How many plants would be in 16 boxes?

Alex’s solution strategy

<table>
<thead>
<tr>
<th>Boxes</th>
<th>1</th>
<th>2</th>
<th>4</th>
<th>8</th>
<th>16</th>
</tr>
</thead>
<tbody>
<tr>
<td>Plants</td>
<td>35</td>
<td>70</td>
<td>140</td>
<td>280</td>
<td>560</td>
</tr>
</tbody>
</table>

Sasha’s solution strategy

<table>
<thead>
<tr>
<th>Boxes</th>
<th>1</th>
<th>10</th>
<th>2</th>
<th>6</th>
<th>16</th>
</tr>
</thead>
<tbody>
<tr>
<td>Plants</td>
<td>35</td>
<td>350</td>
<td>70</td>
<td>210</td>
<td>560</td>
</tr>
</tbody>
</table>

Summary

The development of proportional reasoning is a gradual process, underpinned by increasingly more sophisticated multiplicative thinking and the ability to compare two quantities in relative (multiplicative) rather than absolute (additive) terms. Proportional reasoning as part of the multiplicative field has been identified as a key concept underlying a wide range of topics studied at the middle-school level. The task for middle-years teachers is to assist students to build, consolidate and link their proportional reasoning ability; not an easy task, as research consistently indicates students’ difficulty with proportion related topics. Our goal in this project was to explore and develop ways to support students’ proportional thinking, and to assist them to make connections between core ideas within mathematics and science.

Acknowledgements

Some material contained in these notes has been taken and adapted from Lamon (2005) (see reference list for complete citation). This includes:

The 10 questions in the “Thinking about proportion” section

The 10 questions included in the “Further thinking about proportional reasoning”

The description of absolute and relative thinking with respect to the crocodiles

The example in the “Additive to multiplicative thinking” about Ruby and Trent and their tazos, the pizza, and the description of Sally’s week

The four items for promoting discussion on additive and multiplicative thinking (eggs, Dan & Tasha, the rectangle, Fred’s comment)

Notes on inverse proportion

The cycling track situation is an adaptation of the oft-cited running track problem presented in Cramer, Post and Currier (1988).

The ratio tables examples have been taken from the National Science Foundation (1998) materials.
References and further reading


Proportional Reasoning Activities
The Empty Number Line

The empty number line can be used to promote multiplicative thinking and number sense. Using a piece of washing line, some pegs and some cut-out numbers can be used to enhance students’ understanding of whole numbers, decimals, fractions, percentages.

1. Get two volunteers to hold the line tight.
2. Peg the zero card at one end and the five card at the other.
3. Get another volunteer to peg the number 2 card on the number line.
4. Ask the class to determine how a check could be done to find out the accuracy of the position of the 2 card (fold the zero position to the two position and halve to find the 1 position, then fold five times to see how close to the pegged five this position is. Students need to be able to see that there are five equal intervals between the zero and the five).

Where are they?

Try the following combinations:

<table>
<thead>
<tr>
<th>Peg:</th>
<th>Find:</th>
</tr>
</thead>
<tbody>
<tr>
<td>6 and 10</td>
<td>9</td>
</tr>
<tr>
<td>5 and 14</td>
<td>7</td>
</tr>
<tr>
<td>50 and 80</td>
<td>70</td>
</tr>
<tr>
<td>0 and 2</td>
<td>$1^{2/3}$</td>
</tr>
<tr>
<td>1 and 2</td>
<td>$1^{2/3}$</td>
</tr>
<tr>
<td>2 and 3</td>
<td>$2^{1/4}$</td>
</tr>
<tr>
<td>0 and 1</td>
<td>0.6</td>
</tr>
<tr>
<td>2 and 4</td>
<td>3.5</td>
</tr>
<tr>
<td>0 and 1.5</td>
<td>1.25</td>
</tr>
<tr>
<td>0 and 1</td>
<td>25%</td>
</tr>
<tr>
<td>0 and 1</td>
<td>80%</td>
</tr>
<tr>
<td>0 and 1</td>
<td>30%</td>
</tr>
</tbody>
</table>

Ordering numbers

Place the following number sets in the correct place on the number line

- 2, 1.5, $\frac{3}{4}$
- $\frac{3}{4}, 2\frac{1}{2}, 0.6$
- 20%, 2, 0.8
- 25%, $\frac{1}{4}, 0.25, 2$
- $\frac{1}{8}, 1\frac{1}{4}, \frac{3}{6}$
**Visualising number lines**
Imagine a number line, and locate the numbers 1 and 2
- now locate 0
- now locate -1 and -2
- now locate 23
- now locate 101
- now locate -99
Where are these numbers?
What can you say about numbers? (infinite)

**Moving around the number line**
- What number is three to the right of 5?
- What number is six to the left of 4?
- What number is twice as far from 3 as from -7? (there are two possible answers: 23 or -17)
- What number is half as far from 9 as from 5? (there are two possible answers: 11 or 7)
- What number is three times as far from 6 as from -1? (there are two possible answers: 27 or -15)
- What number is one-fifth as far from -5 as 10? (There are two possible answers: -2 or -8)

**Conceptualising very small numbers**
Locate the half-way point between 1 and 2.
What is its decimal name? (1.5)

Focus on the point whose decimal name is 1.6
How far apart are 1.5 and 1.6? (one-tenth)
What about the point half-way between 1.55 and 1.56? (1.555)

Draw and redraw the number line to show these points.
What is the decimal name of the point one-tenth of the way from 1 to 2? (1.1)
What is the decimal-name of the point one-tenth of the way from 1.1 to 1.2? (1.11)
What is the decimal-name of the point one-tenth of the way from 1.11 to 1.12? (1.111)
What is the decimal-name of the point one-tenth to the right of 1? (1.1)
What is the decimal-name of the point one-hundredth to the right of that? (1.01)
What is the decimal-name of the point one-thousandth to the right of that? (1.001)

Keep going until you have established a pattern and can explain it to colleagues.

Locate the following numbers on a number line. How far apart are each pair?
- 1.7 and 1.8
- 1.73 and 1.74
- 1.732 and 1.733
- 1.7320 and 1.7321

These exercises help to establish a feel for the magnitude of decimal numbers, the infinite numbers that can be located on the number line, the fluid nature of the number line, the insignificance in the decimal places beyond thousandths.
How big is each state/territory in Australia?


1. In your mind, visualise a map of Australia.
2. Locate Victoria.
3. Think of the size of Victoria in relation to all other states/territories in Australia.
4. Create a data table as indicated:

<table>
<thead>
<tr>
<th>State/Territory</th>
<th>1st Try</th>
<th>2nd Try</th>
<th>Actual</th>
<th>Area: km²</th>
</tr>
</thead>
<tbody>
<tr>
<td>Vic</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>NSW</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Qld</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>SA</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>WA</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>NT</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Tas</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Act</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

5. Give the size of Victoria a unit of 1. Using Victoria as a reference point, estimate the size of each of the other states/territories in relation to Victoria. That is, how many times bigger/smaller than Victoria is each other state/territory. Write your estimate in the 1st Try column. (Do not attempt to locate a map of Australia for assistance at this point).

6. Share your estimates with others around you, but don’t feel the urge to alter your estimate.

7. Now, consult a map of Australia.

8. In the 2nd Try column, write your revised estimate of each of the other states/territories, still using the size of Victoria as a unit of one.

9. Share your estimates with others around you.

10. Complete column three Actual in the data table when you have been given the actual size of each state/territory compared to Victoria (information next page).

11. In the final column Area: km², write in the size of each state in square kilometres.

12. Here is a web address of a site that will give you details of the size of each of the states/territories of Australia:

<table>
<thead>
<tr>
<th>State/Territory</th>
<th>1&lt;sup&gt;st&lt;/sup&gt; Try</th>
<th>2&lt;sup&gt;nd&lt;/sup&gt; Try</th>
<th>Actual</th>
<th>Area: km²</th>
</tr>
</thead>
<tbody>
<tr>
<td>Vic</td>
<td>1</td>
<td></td>
<td></td>
<td>227 600</td>
</tr>
<tr>
<td>NSW</td>
<td></td>
<td>3.5</td>
<td></td>
<td>801 600</td>
</tr>
<tr>
<td>Qld</td>
<td></td>
<td>7.6</td>
<td></td>
<td>1 727 200</td>
</tr>
<tr>
<td>SA</td>
<td></td>
<td>4.3</td>
<td></td>
<td>980 000</td>
</tr>
<tr>
<td>WA</td>
<td></td>
<td>11.1</td>
<td></td>
<td>2 525 500</td>
</tr>
<tr>
<td>NT</td>
<td></td>
<td>5.9</td>
<td></td>
<td>1 346 200</td>
</tr>
<tr>
<td>Tas</td>
<td></td>
<td>0.3</td>
<td></td>
<td>67 800</td>
</tr>
<tr>
<td>Act</td>
<td></td>
<td></td>
<td></td>
<td>2400</td>
</tr>
</tbody>
</table>
Why do penguins huddle?

In cool arctic conditions, penguins tend to huddle together. The huddle is in a constant state of motion, with the penguins on the outer of the huddle shuffling to get into the centre of the huddle. And the penguins in the centre of the huddle constantly being pushed around trying to maintain their position in the centre of the huddle. Why do they do this? The simple explanation is that they are trying to get warm. But a more scientific explanation is when you consider the conditions that a huddle creates.

In this investigation, you will create a huddle of penguins and explore features of the huddle by varying the number of penguins in it. The investigation uses cubes to represent penguins.

Materials: 2 cm wooden or plastic blocks, calculator

1. Using 3 cubes, construct a 3-dimensional penguin in an upright position (one cube for the head, one cube for the upper body, one cube for the lower body).
2. Determine the volume of the penguin (count the cubes)
3. Determine the surface area of the penguin (count the number of square faces that are on the outside of the penguin).
4. Construct a data table, and enter this data in the data table.
5. Calculate the surface area to volume ratio.

<table>
<thead>
<tr>
<th>Number of Penguins</th>
<th>Surface Area</th>
<th>Volume</th>
<th>Ratio SA:V</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

6. Construct more penguins and completely surround the first penguin with the new penguins. You should have a total of 9 penguins in the huddle this time.
7. Calculate the surface area, volume and determine the ratio.
8. Create more penguins and determine the surface area, volume and ratio.
9. Examine the data and discuss what is happening as the number of penguins in the huddle increases. What can you say about the surface area and volume ratio as the huddle increases?
10. Write a statement explaining why penguins huddle, making mention of the ratio between surface area and volume?
11. Are there any other instances when surface area and volume are an issue?

Going further
12. Can you think of why babies and small animals suffer much more in hot cars than adults? Why do chips fry faster than whole potatoes?
Exploring the Golden Ratio

Materials: copies of 3 rectangles (master attached), rulers (mm), calculators

1. Distribute copies of the three rectangles to each student, face down on the desk. Tell the students that, on your signal, they are to turn over their piece of paper and glance quickly at the three rectangles displayed, picking out the one that catches their eye; the one that they like the most.

2. On the board, record the number of students who picked Rectangle A, Rectangle B and Rectangle C. Leave this information on the board for later.

3. Ask students to accurately measure the sides of each rectangle in mm (record on the sheet). Record these measures on the board in a ratio table form so that students can copy onto their paper. Record the longest side and the shortest side for each, and include the appropriate measures.

4. Make the shortest side measure a value of 1, and show the operation of division. Indicate on the ratio table the value of the longest side when the shortest side is ‘1’. Then, make the longest side a value of ‘1’, and find the value of the shortest side when the longest side has a value of ‘1’. (See below)

<table>
<thead>
<tr>
<th>Rectangle A</th>
<th>Shortest Side</th>
<th>Longest Side</th>
<th>+ 11</th>
<th>÷ 11</th>
<th>÷ 46</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>11</td>
<td>46</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Rectangle B</th>
<th>Longest Side</th>
<th>Shortest Side</th>
<th>÷ 28</th>
</tr>
</thead>
<tbody>
<tr>
<td>28</td>
<td>45</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Rectangle B</th>
<th>Longest Side</th>
<th>Shortest Side</th>
<th>÷ 45</th>
</tr>
</thead>
<tbody>
<tr>
<td>28</td>
<td>45</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>
5. Discuss how to interpret each of the new measurements in terms of the relationship between the width and the length. That is, we have made the shortest side a unit of ‘1’. In the first rectangle, the ratio of the shortest side to the longest side is 1:4.18. The longest side in Rectangle A is 4.18 times the size of the shortest side.

The ratio of 1:1.6 or 1:0.6 for a rectangle (depending whether the width or the length is your focus) is known as the **Golden Ratio**. Refer back to selection of rectangles at the beginning of this activity. How many students chose the rectangle that was in the Golden Ratio? Why? Is it the most appealing rectangle to the human eye? Is this cultural?

---

**The Golden Ratio**

Rectangles that have the length and width in a golden ratio are said to be **Golden Rectangles**. Golden rectangles are said to be the most pleasing rectangle to the human eye. Examples of golden rectangles are found in art and architecture throughout the centuries. The Mona Lisa, painted by Leonardo da Vinci, is actually framed within a golden rectangle.

One of the most interesting facts about the golden rectangle is that the Parthenon in Greece was built in the 5th century BC and fits into a golden rectangle, even though its builders would have had no knowledge of the golden rectangle.
Finding Golden Rectangles

Find some rectangles in the room. Measure and record the length and width of each rectangle. Use your calculator to find out whether your rectangle is golden (make the shortest side the unit ratio, i.e., make it ‘1’). Record what you found by completing the columns on the board:

<table>
<thead>
<tr>
<th>Object</th>
<th>Shortest Side</th>
<th>Longest Side</th>
<th>Ratio S:L</th>
<th>Golden? Yes/No</th>
</tr>
</thead>
<tbody>
<tr>
<td>Desk</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Board</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Light fitting</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Light switch</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

1 : 1.618

Constructing Golden Rectangles

Using a ruler and your knowledge of the golden ratio, construct some golden rectangles. What was the process required to make a range of rectangles in the golden ratio? Discuss the strategies used by various people in the room. Make a collage of golden rectangles.
Exploring the Fibonacci Sequence

The Fibonacci sequence is the following: 1, 1, 2, 3, 5, 8, 13, 21…

What is the pattern in this sequence? Continue the next 5 numbers in the sequence.

Explore the relationship between the numbers in the Fibonacci sequence by constructing a table like the one below:

<table>
<thead>
<tr>
<th>numbers</th>
<th>ratio</th>
<th>decimal</th>
</tr>
</thead>
<tbody>
<tr>
<td>2, 1</td>
<td>2:1</td>
<td>0.5</td>
</tr>
<tr>
<td>3, 2</td>
<td>3:2</td>
<td>1.5</td>
</tr>
<tr>
<td>5, 3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8, 5</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Using a calculator, divide the second number of each pair into the first number. Record this in the third column of your table. As you continue to do this with pairs of numbers in the sequence, what do you notice about the ratio between the two numbers? Spooky!

What happens if you divide the other way?

Construct a new table like the first one, but divide the other way and record this number in the third column. What happens?

<table>
<thead>
<tr>
<th>numbers</th>
<th>ratio</th>
<th>decimal</th>
</tr>
</thead>
<tbody>
<tr>
<td>1, 2</td>
<td>2:1</td>
<td>0.5</td>
</tr>
<tr>
<td>2, 3</td>
<td>3:2</td>
<td>1.5</td>
</tr>
<tr>
<td>3, 5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5, 8</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
## Ratio tables

One way of assisting students to develop mental strategies for solving proportion problems is through the use of ratio tables. Ratio tables are a convenient way of symbolising the elements within proportion situations, and for supporting thinking strategies for solution. Ratio tables encourage the use of number strategies such as halving, doubling, multiplying by 10, and so on. Some examples of ratio tables and how they can be used for solving proportion situations are below:

### Example 1

There is 1 tent per 12 children. How many children can fit in 14 tents?

<table>
<thead>
<tr>
<th>Tents</th>
<th>1</th>
<th>10</th>
<th>5</th>
<th>15</th>
<th>14</th>
</tr>
</thead>
<tbody>
<tr>
<td>Children</td>
<td>12</td>
<td>120</td>
<td>60</td>
<td>180</td>
<td>168</td>
</tr>
</tbody>
</table>

Strategy: times by 10, divide by 2, multiply by 3, subtract 1 group of children.

### Example 2

Bottles of water are packaged into 15 bottle boxes. How many bottles of water would there be in 16 packages?

<table>
<thead>
<tr>
<th>Package</th>
<th>1</th>
<th>10</th>
<th>5</th>
<th>15</th>
<th>16</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bottles</td>
<td>15</td>
<td>150</td>
<td>75</td>
<td>225</td>
<td>240</td>
</tr>
</tbody>
</table>

Strategy: times by 10, divide by 2, multiply by 3, add 1 group of bottles.

### Add notes to the arrows to indicate the strategy

Some students’ solution strategies

Seedling plants come in boxes of 35 plants. How many plants would be in 16 boxes?

Alex’s solution strategy

<table>
<thead>
<tr>
<th>Boxes</th>
<th>1</th>
<th>2</th>
<th>4</th>
<th>8</th>
<th>16</th>
</tr>
</thead>
<tbody>
<tr>
<td>Plants</td>
<td>35</td>
<td>70</td>
<td>140</td>
<td>280</td>
<td>560</td>
</tr>
</tbody>
</table>
Sasha’s solution strategy

<table>
<thead>
<tr>
<th>Boxes</th>
<th>1</th>
<th>10</th>
<th>2</th>
<th>6</th>
<th>16</th>
</tr>
</thead>
<tbody>
<tr>
<td>Plants</td>
<td>35</td>
<td>350</td>
<td>70</td>
<td>210</td>
<td>560</td>
</tr>
</tbody>
</table>

*Try some for yourself*

Juice is packaged in cases of 14 bottles. How many bottles of juice are in 9 cases?

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</tbody>
</table>

At school, 98 bottles of juice were required. How many cases needed to be bought?

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</tbody>
</table>

Mangoes are 2 for $3. How many could you buy for $7.50?

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<th></th>
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</thead>
<tbody>
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</tbody>
</table>

Apples are 3 for $2. How much would 33 apples cost?

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<table>
<thead>
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<th></th>
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</thead>
<tbody>
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</tr>
</tbody>
</table>

Sun City: Population 625, Horses 125
Moon City: Population 720; Horses 120
Dunstown: Population 1368; 28 horses per 100 inhabitants

1. How many horses per inhabitant in Sun City?
2. How many horses per resident in Moon City?
3. Which town has relatively the smallest number of horses: Sun City, Moon City or Dunstown?
4. How many horses in Dunstown?
Balloon Rockets

Materials
- round balloons
- sticky tape
- scissors
- fishing line
- drinking straws
- tape measure (long one)
- stop watches

1. Take a length of fishing line (approximately 15 metres long) and tie it to a secure structure (e.g., the back of a chair, the balcony rail, the door knob).
2. Thread a drinking straw through the fishing line.
3. Blow up a balloon, and insert a 4 cm length of straw in the end and fix with sticky tape, holding firmly so that the air does not escape.
4. Tape the balloon to the drinking straw on the fishing line, still holding the end of the balloon to ensure that no air escapes.
5. Whilst holding the balloon, extend the fishing line its full length until it is taut.
6. On the signal (i.e., when the stop watch timer says “Go!”), let the balloon go and time it until it stops.
7. Measure the distance the balloon travelled.
8. Construct a table to record your results:

<table>
<thead>
<tr>
<th>Trial</th>
<th>Description</th>
<th>Distance (m)</th>
<th>Time (sec)</th>
<th>Speed</th>
<th>Thoughts</th>
</tr>
</thead>
<tbody>
<tr>
<td>T1</td>
<td>1 straw exhaust</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>T2</td>
<td>2 straw exhaust</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>T3</td>
<td>3 straw exhaust</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>T4</td>
<td>4 straw exhaust</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>T5</td>
<td>5 straw exhaust</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
9. Take a new balloon and repeat the exercise, this time taping two 4 cm straws into the end. Before you let the balloon go, predict what will happen. Record the distance travelled in the time, and enter into the data table.

10. Repeat three more times, inserting 3, then 4 and then 5 straws into the end of the balloon.

11. As you observe each balloon, write your thoughts about the balloon in the “Thoughts” column of your table.

12. Analyse the data table. What can you say about the number of straws in the balloon and the speed and distance travelled?

**Rocket modifications**
What modifications could you make to your rocket to alter the speed, or distance? Try using different length exhausts; altering the length of the fishing line; methods of taping the balloon to the straw. Is there an ideal exhaust? Experiment, record, analyse.

**Sumo Wrestlers**
From your experimentation with balloon rockets, create a balloon that will take part in a sumo wrestling match with another balloon. For the tournament, your sumo wrestler will challenge another sumo wrestler:

1. Construct a tournament line with a piece of fishing line, tying the end to something solid.
2. Place two drinking straws on the line.
3. Tape your sumo wrestler onto one of the straws, and your challenger will tape his/her balloon to the other straw. At the signal, let the balloons go. The balloon that can move the other balloon the longest distance is the winner.

**The Mathematics of Tournaments**
Explore the mathematics of organising a tournament:
- What is the mathematics involved in organising the tournament?
- What rules will be used for eliminating opponents?
- What is the ideal number of entrants for a tournament?
- What if there are 13 entrants?
- What if there are 16 entrants?
- Will there be finals?

**The mathematics and science of speed**
- This investigation provides rich data to use to promote rich, conceptual understanding of the concept of speed as distance divided by time. By plotting graphs and discussing the slopes of lines, students will develop intuitive notions about gradient. They will be able to predict the types of situations that graphs depict, depending upon the slope of the line.
Mass Volume Density

Activity 1 - Exploring Volume

Materials:
- Set of 6 jars (labelled A-F) of same size
- Weighing scales
- Water bath
- Tape measure
- Cloth (for wiping up spills)
- Ruler
- Tea towel (for wiping jars)
- Pen and paper
- Large measuring jug (marked in mL)
- Calculator
- Activity 1 Worksheet

4. Without using the weighing scales, arrange the jars in order of heaviness by heft (lifting them and feeling how heavy they are). Write down the order. It is possible that your group will not agree on the order.

5. Using the weighing scales, find the mass of each container and reorder the jars. Record the mass of each jar.

6. Using the tape measure and/or ruler, make the following measurements of one of the jars (record these measurements on your page):
   - Diameter of circle at base of jar:
   - Radius of circle (diameter ÷ 2):
   - Height of jar:

7. Calculate the volume of the jar and record it on your page. Volume means the amount of space something takes up. To find the volume of many 3D shapes, use the formula: area of base x height. The base of this jar is a circle

8. Find the volume of one of the jars by using the displacement method (record this measurement on your page): Put the jar into a measuring jug of water. Note the water level increase.

9. Discuss the following questions in your group (complete questions on your page): Did you get a similar result for volume using both the calculation and the displacement method? Should these two measures be the same? Why/why not?

10. How would you describe the volume of each of the jars in the set? What can you say about the mass of each of the jars in the set?

11. Think about the meaning of the word density: Density is the relationship between the mass and volume of the object. All the jars in this activity have density (just as they also all have volume and mass). Density means considering both mass and volume at the same time.

12. Which jar has the greatest density? Which jar has the least density? Why do you think so? The second activity is to assist you to do this.
**Activity 1 Worksheet - Exploring Volume**

<table>
<thead>
<tr>
<th>Jar</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mass</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Diameter of Base: __________________

Height: __________________

Radius of Base: __________________

Area of Base: __________________

Volume of jar: __________________

(area of base x height)

Volume by displacement: ______________

**Questions:**

1. What can you say about the volume of the jar using the calculation method compared to the displacement method?

__________________________________________________________________

2. Should the volume measure be the same using either of these methods? Explain.

__________________________________________________________________

3. Complete the following statement:

*Although the jars have the same __________, they all have a different __________*_

4. Describe how to find volume by calculation:

__________________________________________________________________

__________________________________________________________________

5. Describe how to find volume by displacement:

__________________________________________________________________

In this activity you explored the volume and mass of the jars. Another property that can be explored is the density of each jar. All the jars in this activity have density (just as they also all have volume and mass). Density is the relationship between the mass and volume of the object.

Discuss: Which jar has the greatest density? Which jar has the least density? Why do you think so?
Activity 2 - Exploring Density

Materials:
- Set of 5 jars (labelled 1-5) of different sizes
- Weighing scales
- Water bath
- Tape measure
- Cloth (for wiping up spills)
- Ruler
- Tea towel (for wiping jars)
- Pen and paper
- Large measuring jug (marked in mL)
- Calculator
- Activity 2 Worksheet

1. Order the jars from largest to smallest volume.
2. Discuss with your group whether you need to calculate volume to complete the first task and explain why or why not.
3. Without using the weighing scales, arrange the jars in order of heaviness by heft (lifting them and feeling how heavy they are). Write down the order. It is possible that your group will not agree on the order.
4. Using the weighing scales, find the mass of each container and reorder the jars. Record the mass of each jar.
5. Calculate the volume of each jar. Use the measuring tape or ruler to take appropriate measurements. Enter the information in the data table on the worksheet.
6. Find the volume of each of the jars by using the displacement method. Record the information in the data table on the worksheet.
7. Discuss the accuracy of your volume measurements using both methods.
8. Complete Data Table 2 on the worksheet. Calculate density by dividing mass by volume.
9. Discuss the following questions in your group:
   - Which jar has the greatest density?
   - How do you know?
   - What do you need to consider when you are thinking about density?
10. By referring back to your calculations in Activity 1, complete Data Table 3 and calculate the density of jars A-F.
11. Refer to the data in Tables 2 and 3 and order the jars according to their density.
12. Discuss with your group the meaning of the word density. Write your definition of density on your worksheet.
**Activity 2 Worksheet - Exploring Density**

### Data Table 1 - Volume and Mass of Jars 1-5

<table>
<thead>
<tr>
<th>Jar</th>
<th>Mass</th>
<th>Diameter of base</th>
<th>Radius</th>
<th>Area of base</th>
<th>Height</th>
<th>Volume (Area of base x height)</th>
<th>Volume by displacement</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
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<tr>
<td>4</td>
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<tr>
<td>5</td>
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</tr>
</tbody>
</table>

### Data Table 2 - Density of jars 1-5

<table>
<thead>
<tr>
<th>Jar</th>
<th>Mass</th>
<th>Volume (approx)</th>
<th>Density Mass/Volume</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>3</td>
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<td></td>
<td></td>
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<tr>
<td>4</td>
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<tr>
<td>5</td>
<td></td>
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<td></td>
</tr>
</tbody>
</table>

### Data Table 3 - Density of jars A-F

<table>
<thead>
<tr>
<th>Jar</th>
<th>Mass</th>
<th>Volume (approx)</th>
<th>Density Mass/Volume</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>B</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>C</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>D</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>E</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Activity 2 Worksheet
(Continued)

List the jars in order of increasing density:

________________________________________________________________

Density: (Write your group’s definition of density here)

________________________________________________________________

________________________________________________________________

________________________________________________________________

________________________________________________________________

38
**Activity 3 - Sinking and Floating**

**Materials:**
- Set of 6-8 jars of various volumes and masses
- Weighing scales
- Water bath
- Tape measure
- Cloths (for wiping up spills)
- Ruler
- Tea towel (for wiping jars)
- Pen and paper
- Large measuring jug (marked in mL)
- Calculator
- Activity 3 Worksheet

1. Find the volume and mass of each of the jars and enter this information in the data table on the worksheet. Calculate the density of each jar.
2. For each jar, predict whether it will sink or float in water. Record your prediction in the data table on the worksheet.
3. Test whether your prediction was correct. Enter this information in the table on the worksheet.
4. Analyse the results of the data table. How can you predict whether something will sink or float in water? Write your reasoning on the worksheet.
5. Complete the questions on the worksheet.
### Activity 3 Worksheet - Sinking and Floating

#### Data Table 1 - Does it sink or float?

<table>
<thead>
<tr>
<th>Jar</th>
<th>Mass</th>
<th>Volume (approx)</th>
<th>Density Mass/Volume</th>
<th>Sink or Float? Prediction</th>
<th>Sink or Float? Actual</th>
</tr>
</thead>
<tbody>
<tr>
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</tr>
</tbody>
</table>

#### Questions

1. How can you predict whether something will sink or float in water? Refer to examples in your table to support your statement:

   _______________________________________________________________
   _______________________________________________________________
   _______________________________________________________________

2. Jar X weighs 500g and has a volume of 400 cm³. It sinks in water. Jar Y weighs 600 g and has a volume of 700 cm³. It floats on water. For each of the following jars, say whether they will sink or float in water and why you think so:

   a) Mass 500g, volume 300 cm³.
      - sink
      - float
      Reason: ________________________________________________________

   b) Mass 500g, volume 750 cm³.
      - sink
      - float
      Reason: ________________________________________________________

   c) Mass 250g, volume 200 cm³.
      - sink
      - float
      Reason: ________________________________________________________
d) Mass 1500g, volume 1800 cm³.
  □ sink
  □ float
Reason: ...........................................................................................................

e) Mass 900g, volume 1200 cm³.
  □ sink
  □ float
Reason: ...........................................................................................................

3. A model ship weighs 3000 g. If it is to float, what can you say about its total volume?
....................................................................................................................
....................................................................................................................
....................................................................................................................

4. I got a plastic submarine out of my cereal packet this morning. Sometimes it floats and sometimes it sinks. Its total volume is 5 cm³.
(a) What must its mass be to make it sink?
....................................................................................................................
....................................................................................................................
(b) What must its mass be if it is to float?
....................................................................................................................
....................................................................................................................
....................................................................................................................

5.a) What size will the bag of feathers be that would weigh the same as a brick? Explain.
....................................................................................................................
....................................................................................................................

b) Will a kilogram of feathers weigh the same as a kilogram of bricks? Explain.
....................................................................................................................
....................................................................................................................

c) Will a kilogram of feathers have the same density as a kilogram of feathers? Explain.
....................................................................................................................
....................................................................................................................

(d) Which has the greater density, a brick made out of clay or a brick the same size made out of foam? Explain.
....................................................................................................................
....................................................................................................................
**Pocket Money**

You have decided to save $2 per week of your pocket money. Complete the following ratio table to show how your money grows over the weeks.

<table>
<thead>
<tr>
<th>Week</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>10</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Amount</td>
<td>$2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Using the information in the ratio table, draw a graph to show how your money grows each week.

What if you decide to save $4 per week? Complete the ratio table to show this.

<table>
<thead>
<tr>
<th>Week</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>10</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Amount</td>
<td>$4</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Use a different colour and show how your savings grow at $4 per week on the graph.
**Pocket Money**  
(Continued)

**Saving $1 per week**  
What if you decide to save only $1 per week? Complete the ratio table to show this.

<table>
<thead>
<tr>
<th>Week</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>10</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Amount</td>
<td>$1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Using a different colour, draw this on your graph.

**Saving $7 per week**  
What if you decide to save $7 per week? Complete the ratio table to show this.

<table>
<thead>
<tr>
<th>Week</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>10</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Amount</td>
<td>$7</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Using a different colour, draw this on your graph.

**What can you see from your graphs?**

1. Look at your graph for saving $2 per week. Put a * on that graph to show how much money you would have after 7 weeks.

2. Look at the graph for saving $1 per week. Put a * in a different colour on that graph to show how much money you would have after 7 weeks.

3. Look at the two stars. Write all the things that you can say about why the stars are at those positions.

4. What do you notice about the graph when you save $1 per week compared to saving $4 per week? Write down as many things as you can.

________________________________________

________________________________________

________________________________________

________________________________________

________________________________________
Tell me a story

What do you think this graph is telling a story about?
Label each axis to match your story
Creating Stories

Make up two different stories for the following two graphs. Remember to label the axes to match your stories. Be creative and imaginative.
Fractions and Decimals

What are fractions?
Fractions are numbers, and belong to the set of rational numbers. Decimals are also rational numbers. Whole numbers, counting numbers and integers (positive and negative numbers) are also a subset of the numbers we call rational.
Fractions can be located on a number line, and are generally thought of as parts of a whole.

Fraction comparisons
A lot of fraction comparison can occur through evoking mental images. For example, consider the following fraction pairs, and think about what came into your head when you were trying to compare them.
Which is bigger:
   a) $\frac{1}{3}$ or $\frac{1}{5}$?
   b) $\frac{3}{4}$ or $\frac{5}{6}$
   c) $\frac{2}{9}$ or $\frac{4}{9}$
   d) $\frac{2}{3}$ or $\frac{2}{5}$
   e) $\frac{3}{7}$ or $\frac{5}{9}$
Did some mental pictures come into your head? What were they?

When building students’ conceptual knowledge of fractions, we should always assist them to be able to build strong mental pictures so that they do not have to resort to computation or calculation to determine the size of fractions.

Meanings of fractions
- part of a whole
- part of a set
- division
- ratio
- operator (that which multiplies by numerator and divides by denominator)

Part of a whole fraction meaning
The steps for developing fraction understanding are as follows:
1. identify the whole (“This is one whole”)
2. partition into equal parts (divide into 3 equal parts)
3. name the parts (each part is $\frac{1}{3}$)
4. determine the number of parts (I need 2 parts)
5. name the fraction (this is $\frac{2}{3}$)

“Reversing” - a teaching strategy
When we teach, we must be careful that our teaching is not “one way”. That is, we should not always ask students to perform the same types of operations. For example, we often ask students to partition a “whole” into an equal number of parts. We also should go the other way (reverse), by showing students a model of a particular number of parts, and asking them to construct the whole. Try it! Get a piece of paper, and show me three-quarters. Then take another piece of paper and tell yourself that this piece of paper represents two-thirds. Now create the original whole.

Reversing should not only be integral to teaching fractions, but all other maths topics as well.
Teaching model

As always, we must pay attention to the three modes of representation when we teach any concept - concrete, oral, symbolic. This is particularly important as we include a new fraction name for “fourths” - quarters. We also need to ensure that students know what the two numbers in the symbolic fraction representation stand for. For example, in the fraction \( \frac{3}{4} \), the bottom number (denominator) tells us how many parts the whole has been divided into, and the top number (numerator) tells us how many parts we have taken.

Fraction models

(i) Region
   e.g., common 2D shapes - circle, square, rectangle, triangle, etc
   need to show students a variety of these shapes so that they don’t see that “piece of a pie” all the time.

(ii) Set
    e.g., a collection of objects - counters, blocks, etc

(iii) Length
     e.g., string, paper strips, leading to number line

(iv) Area
    These models are similar to region models, but the whole is divided into equal parts that have the same area, but not necessarily the same shape.

Real world fraction models

Take care with using real world fraction models (cakes, pies, pizzas, licorice) as they are rarely divided into equal pieces in the real world.

Sequencing of activities

(i) Partitioning - the whole is cut into parts (sharing)
(ii) Words - number of equal parts; halves, thirds, fourths/quarters
(iii) Counting - one half, two halves - one whole...; one third, two thirds, three thirds - one whole, one and one third...
(iv) Drawing a model - rectangles are easiest to partition
(v) Extending a model - to show equivalent fractions. E.g., draw a diagram to show two thirds; then show four sixths on the same diagram.

Developing part of a whole conceptual understanding

Provide students with many opportunities to divide different models into parts (and also parts back to wholes). Consider the positive and negatives of various models, for example, the rectangle is easy to partition, but students can lose sight of the whole as they are working with the different “bits” of paper; circles are easiest to reconstruct and focus on the whole, but are most difficult to partition into any fractions other than halves and quarters.

Fraction circles

Make two sets of fraction material using the circles printed on white paper, and the 5 sheets of coloured paper.
Sixths

Cut out one white circle and 2 other circles of different colours.

Make one circle into two halves, and write the name of each piece on each piece in words on one side (one third), and in symbols on the other ($\frac{1}{3}$). Do this for the other coloured circle, this time making the pieces into sixths.

Use the fraction pieces to explore various part/whole fraction actions, for example:
- How many thirds make one whole?
- How many more pieces to make one whole if I have two thirds?
- How many wholes if I have 4 thirds?
- What is 2 thirds and 2 thirds?
- How many sixths in one third?
- How many sixths in two thirds?
- How many sixths in one whole?

Make up other simple equivalences, additions and subtractions that can be carried out using the fraction set. Describe the value for children in manipulating such sets of material (and making them for themselves). Discuss the mental pictures that such materials can develop. Discuss the value of the circle as a model.

Eighths

Cut out one white circle and 3 other circles of different colours.

Make one circle into two halves, and write the name of each piece on each piece in words on one side (one half), and in symbols on the other ($\frac{1}{2}$). Do this for the other two coloured circles, this time making the pieces into quarters and eighths respectively.

Explore this set of fraction material in the same way as you did for the sixths. Would it be necessary to have children create fraction sets for all denominators up to 10? Why/why not?
**Fraction strips**

Make a set of fraction strips for tenths, fifths, halves and one whole (see below).

Use different coloured paper for each fraction size.

Write the names on each piece.

Do similar activities as for the fraction circles.

Discuss the value of this model.

Discuss the value of this exploration in terms of its link to decimals.

```
<table>
<thead>
<tr>
<th>one whole</th>
</tr>
</thead>
<tbody>
<tr>
<td>one half</td>
</tr>
<tr>
<td>one fifth</td>
</tr>
<tr>
<td>one tenth</td>
</tr>
</tbody>
</table>
```

**Addition and subtraction of fractions**

Encourage mental images; work with fraction pieces so that students see that there are, for example, 4 quarters in a whole; 5 quarters mean that there is one whole and one quarter left over; that one half is the same as two quarters; that one fifth is smaller than one third.

For addition and subtraction, students should be able to “see” the additions and subtractions in their head, without having to resort to calculation.

Encourage mental computation, such as

\[1 - \frac{1}{3}\]
\[\frac{3}{5} + \frac{1}{5}\]
\[\frac{3}{4} + \frac{3}{4}\]
\[\frac{3}{7} + \frac{4}{7} = 1\text{ whole}\]

**Part of a set**

Students need to see the connection between fractions of one whole, and fractions of a set. They need to see that the process is essentially the same, and that the entire set is the “whole”. This cannot be over-emphasised enough as many students have difficulty with determining such calculations as one third of 6; two-fifths of 20, and the like.

The teaching sequence for part of a set is the same as for part of a whole, and must begin with students focusing of the whole. Placing the objects on a piece of paper, or in a circle of string can assist this beginning point.

Using counters, practise finding the *part* given a *whole* (e.g., get 12 counters and find one third). Conversely, practise finding the whole, given a part (e.g., put out 4 counters, and note that this represents one quarter. Add counters so that you have the whole). Think about the cognitive levels required for each activity. Sequence the level of difficulty. Make up further examples according to difficulty level.
Examples:

12 cubes is one whole; show 1/3.
18 cubes is one whole; show 1/6.
20 cubes is one whole; show ¼.
15 cubes is one whole; show 2/3

4 cubes is ¼; show one whole
3 cubes is ½; show one whole
2 cubes is 1/5; show one and three fifths

20 cubes is one and one quarter; show one whole
15 cubes is one and two thirds; show one whole
21 cubes is one and three quarters; show one whole
10 cubes is two and a half; show one whole

If students are operating at an computational level, they may find the following activities more of a challenge. Those students who were operating at a conceptual level will have more chance of finding the solution. Working with the counters to act out the solution process will assist in reaching the answer.

If you have a set of counters that totals 16, and you are told that the 16 is the same as one and a third, can you tell me how many counters would make up one quarter? (Hint: the answer is not 4).

Here are some more of this type:

20 cubes is one and two-thirds; show one quarter
15 cubes is one and a half; show two fifths
20 cubes is one and a quarter; show two and a half
24 cubes is one and one fifth; show three quarters
21 counters is one and three quarters; show two thirds
24 counters is one and one third; show one half

Here is a real world application of the knowledge required for the above situation:

A farmer bought a small farm. To do this, he borrowed 3/4 of the price, interest-free, from his father in law. Two years later, he sold the farm for 2/3 more than he paid for it. The selling price was $200 000. How much did he have to repay his father-in-law.
Fractions as division

Consider the problem 3 divided by 5. Put it into a real context, and see what happens. E.g., share 3 pizzas between 5 people. Take 3 circles and discuss how the three pizzas could be cut up and shared. If you do this by cutting each circle into fifths, and sharing the fifths between the 5 people, you will see that each person gets three fifths. So what has this shown? That 3 divided by 5 is the same as three fifths. This action shows that the operation is division and that it is partition division (sharing, dealing).

Students need explicit teaching to develop the concept of fractions as division. They need to see that a fraction written in symbolic form means the top number divided by the bottom number. This is easier to see when we consider improper fractions. E.g., \( \frac{4}{3} \) means 4 divided by 3, and hence the “rule” we learnt that to change an improper fraction into a mixed number, look at the bottom number and say “how many threes go into four”. Keep thinking of the pizza division - for this example, we have 4 pizzas and three people. Each person gets 1 whole pizza and one third of the extra pizza.

Fraction multiplication

How can fraction multiplication be modelled meaningfully? One way to start thinking about this, is to think about whole number multiplication. How do we encourage children to think of 3x4? Do we say 3 groups of 4? Try this for fractions, but select your examples carefully. E.g., 3 x \( \frac{1}{2} \) could be read as :3 groups of one half. What pictures might be drawn to represent this? Try other fraction combinations.

Think about using a number line. When we show 3x4 on a number line, we jump the number line in fours, three times (assuming we read the number sentence as three groups of 4). Could we use this representation for 3 x \( \frac{1}{2} \)? Try it. Draw some pictures; draw some number lines. Encourage students to discuss various situations and solutions with others in your group.

6 x \( \frac{1}{2} \) makes more sense when you say “6 groups of \( \frac{1}{2} \)”. Draw a diagram to show that the solution is 3.

Fraction division

How can fraction division be modeled meaningfully? How do we read 6÷\( \frac{1}{2} \)? How would we read such a situation if we were using whole numbers? For 6÷2ould we say: How many twos in six? Could we say something similar for 6÷\( \frac{1}{2} \)? How many halves in 6? Could we use a number line and see how many halves are in one, and then keep going until we found out how many halves in six?

6 ÷ \( \frac{1}{2} \) makes more sense (and hopefully a mental picture) when we say “How many halves in 6?” A number line also may help. There are two halves in one, so there are 12 halves in 6.

Decimals and fraction knowledge

Developing decimal understanding builds on prior knowledge from (i) numeration knowledge and (ii) fraction knowledge.
Linking fractions knowledge to decimal understanding

Prior experience with fractions provides the basis for conceptualising decimals as fractions. The sequence is as follows:

1. Exploration of halves, fifths and tenths as a family of fractions.
2. Symbolic representation of tenths linked to visual image.
3. Symbolic representation of hundredths and tenths to visual image and number line.

A set of fractions made from paper enables the relationship between halves, fifths and tenths to be explored. When such fractions are created using a linear model, the foundation is laid for visualising decimals on a number line. Encourage manipulation of the fraction pieces until students can determine answers to the following types of questions mentally:

- How many tenths in one whole?
- What is three-tenths and three-fifths?
- How many tenths is the same as one-half?
- What is seven-tenths and nine-tenths?
- What’s another name for fifteen-tenths?

Shading pictures of tenths provides a link between the fraction picture, the symbolic fraction form and the symbolic decimal form.

\[
\begin{array}{c}
\text{\tfrac{3}{10}} = 0.3 \\
\hline
\text{\tfrac{13}{10}} = 1.5
\end{array}
\]

The relationship between tenths and hundreds is seen through creating pictures of various fractions and decimals on 10x10 grids. From images created the relative magnitude of tenths in relation to hundredths is seen. Location of decimals on the number line follows from the pictorial representation.
Numeration knowledge

Decimal numbers differ from whole numbers in the addition of a new series of place-value positions to the right of the ones. These new decimal positions are all fractions. Hence the notion of a fraction underpins decimal numeration.

- imagine a number line, and locate the numbers 1 and 2
- now locate 0
- now locate -1 and -2
- now locate 23
- now locate 101
- now locate -99

Where are these numbers?

What can you say about numbers? (infinite)

Conceptualising very small numbers

- locate the half-way point between 1 and 2. What is its decimal name? (1.5)
- focus on the point whose decimal name is 1.6
- how far apart are 1.5 and 1.6? (one-tenth)
- what about the point half-way between 1.55 and 1.56?
- draw and redraw the number line to show this point

These exercises help to establish a feel for the magnitude of decimal numbers, the infinite numbers that can be located on the number line, the fluid nature of the number line, the insignificance in the decimal places beyond thousandths.

Development of decimal number knowledge

- interference caused by whole number knowledge:
  - zero added to rightmost column does not change total value
  - values decrease as move away from decimal point
  - column names end in "ths", not "s"
  - naming numbers start with tenths, not ones
  - naming sequence (tenths, hundredths...) moves left to right, not right to left
  - reading sequence is tenths, hundredths, thousands

Some students develop "rules" for decimals that are incorrect

- the number with the most decimal places is the larger one: e.g., 3.214 is larger than 3.8 because 214 is greater than 8
- The number with the fewer decimal places is the larger one: e.g., 1.2 is greater than 1.35 because 1.2 has fewer decimal places
Keeping it in Proportion (KIIP)
A test for diagnosing students' proportional reasoning

The following pages include:
• Instructions for teachers
• Coding the test
• KIIP Part A
• KIIP Part B
• KIIP Part C
Administering KIIP
Treat this as you would a normal class assessment activity. If appropriate avoid using the word ‘test’ and stress that this is about finding out what students know and can do to inform future teaching decisions.

There are three parts to this assessment:

- Keeping it in Proportion Part A - 40 minutes
- Keeping it in Proportion Part B - 40 minutes
- Keeping it in Proportion Part C - 15 minutes

Students should be allowed maximum opportunity to engage with the tasks in each part without feeling rushed to complete. For this reason, students should be provided with the specified amount of time for each part. To further maximise students’ success on the tasks, each part should be given on a different day. If two parts of the assessment are to be given on the same day, ensure that an adequate break is provided to avoid test fatigue and disengagement.

Assistance during the tasks
Teachers may read the task to any student with reading problems, scribe an oral explanation for students whose thinking may not otherwise be fairly represented, and/or explain unusual words as required.

Teachers can support students by answering questions without telling them what to do. In particular, teachers should avoid providing so much support that students who may not otherwise have completed the task are able to do so with little understanding of what they are doing or why. The object of the exercise is not that students get the right answer, but that they are given an opportunity to demonstrate what they actually do know and can do largely on their own.

Before using the tasks
To encourage students to provide as much evidence of their mathematical thinking as possible, the Worked Examples (attached) should be discussed with students to ensure they understand what is expected of them when they are asked to show their working. Place the attached examples on an OHT and encourage students to consider what it means to

- show your working and explain your answer in as much detail as possible
- use as much mathematics as you can to support your answer.
WORKED EXAMPLES

**Oranges**

$2.98 per kg

**Calculate the total cost if I buy 4 kg of oranges. Show all your working and thinking.**

Here are four students’ responses to this problem:

<table>
<thead>
<tr>
<th>Student A</th>
<th>Student B</th>
</tr>
</thead>
<tbody>
<tr>
<td>$11.92</td>
<td>3 3</td>
</tr>
<tr>
<td></td>
<td>2.98</td>
</tr>
<tr>
<td></td>
<td>x 4</td>
</tr>
<tr>
<td></td>
<td>11.92</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Student C</th>
<th>Student D</th>
</tr>
</thead>
</table>
| 3 times 4 is 12.
Answer is less than $12.
2.98 is 2c less than 3 dollars.
4x 2c is 8c
Subtract 8c from total of $12
Answer $11.92 | $11.92
Because I timesed it |

Questions to ask students:
The problem asked the students to *show your working and explain your answer in as much detail as possible*. Which one do you think did this best?
Sometimes, the problem asks you to *use as much mathematics as you can to support your answer*. Which one do you think did this best? Could you use more mathematics?
**Keeping it in Proportion**

**5-Point Coding Scheme**

**Overview**
As proportional reasoning derives from many mathematics and science domains (fractions, decimals, ratio, percentages, scale, probability, forces, molarity, speed, density), there is a need to consider students’ solution strategies and not just their capacity to perform calculations. For this reason, a two-digit coding scheme has been developed for the *Keeping it in Proportion* test so that students’ responses to proportion tasks can identify solution strategies and not just correct answers. In this way, the two-digit coding scheme serves as a diagnostic assessment, identifying specific topics of weakness that need to be addressed to thus support further proportional reasoning development.

**Coding Scheme**
Each item on the Keeping it in Proportion test is assigned a two-digit code. The first digit indicates whether the response was correct or incorrect (or unattempted), and the second digit indicates the strategy that the student used in solution attainment, as follows:

- **Digit 1:** There are three digits that can be assigned here: 1, 2 or 0.
  - The assigned digit indicates whether the response was correct (digit 1) or incorrect (digit 2), or whether the item was not attempted (digit 0).

- **Digit 2:** There are five digits that can be assigned here: 1, 2, 3, 4 or 0.
  - The assigned digit indicates whether the strategy used in attaining the solution to the item was multiplicative (digit 1), repeated addition (digit 2), additive (digit 3) or inappropriate (digit 4), or whether no strategy or working was shown (digit 0).

The table below summarises the allocation of the two digits, with further details to assist in assigning the second digit (relating to strategy use) given in the last column of the table.

<table>
<thead>
<tr>
<th>First Digit</th>
<th>Second Digit</th>
<th>Details</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>No response</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>Multiplicative strategy</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>Repeated addition strategy</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>Additive strategy</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>Inappropriate strategy</td>
</tr>
</tbody>
</table>

**Codes of interest**
The codes of interest are as follows:
- 11 – correct solution and sophisticated proportional reasoning
- 12 – correct solution and on the way to sophisticated proportional reasoning
- 21 – incorrect solution possibly due to silly calculation error, but evidence of sophisticated proportional reasoning
- 22 – incorrect solution possibly due to silly calculation error, but on the way to sophisticated proportional reasoning
Keeping it in Proportion
Part A

Instructions:
1. Write the answer to each question in the space provided.
2. Use pen only.
3. Do not erase or white-out your answer. If you want to change your answer, cross out your answer and rewrite underneath.
4. Please explain your answers in as much detail as possible to show your thinking and solution strategy.
5. Calculators are not allowed.
6. Time allowed: 40 minutes.
7. Just do your best, even if you are finding some questions quite difficult. We are really interested in your thinking.

Time Allowed: 40 minutes

Equipment Required: Ruler
1. **Butterflies**

   To feed 2 butterflies the zoo needs 5 drops of nectar per day.

   **How many drops would they need each day for 12 butterflies?**

   **Answer:**

   

   Show all your working and explain your answer in as much detail as possible.


2. **Chance (En)counters**

   You are presented with four paper bags containing black and white counters. The number of counters in each bag is provided in the diagrams below. You are blindfolded, the bags are shaken, and you are asked to select a counter from each bag.

   ![Diagram of bags with counters]

   From which bag do you have the best chance of selecting a black counter?

   *Explain the reason for your choice.*
3. **Shopping Trip**

Mary went to the corner store and only spent \( \frac{1}{3} \) of her money. When she got home, she had $6 left.

How much money did she have at the start of her shopping trip?

**Answer:**

*Explain your thinking.*

4. **Three Cups**

Three cups have different amounts of water and sugar. Cup A is full with 3 lumps, Cup B is half full with 2 lumps and Cup C is one third full with 1 lump.

When the lumps have been stirred in, which is the sweetest?

- [ ] Cup A
- [ ] Cup B
- [ ] Cup C
- [x] They're all the same
- [ ] There is not enough information to be sure

*Explain the reason for your choice.*
5. **Sticky Mess**

A recipe for a sticky mess needs 4 cups of sugar and 10 cups of flour. You decide to make a larger amount of the recipe, and have 6 cups of sugar. So, the situation is this:

4 cups sugar and 10 cups flour for the original recipe
6 cups sugar and ___ cups flour for the larger quantity

**How many cups of flour will you need for the recipe to work?**

*Answer:*

*Show all your working.*


---

6. **Fence Painting**

Six people can paint a fence in 3 days. Assume that all of the people work at the same rate.

**How many people would it take to paint the fence in 2 days?**

*Answer:*

*Show all your working.*


---
7. **End of Term Activity**

A quick survey was conducted of Year 5 and 6 students on their first choice for an end of term activity. The table shows how many students chose each activity.

<table>
<thead>
<tr>
<th></th>
<th>Going to the beach</th>
<th>Going to the movies</th>
<th>Going bowling</th>
</tr>
</thead>
<tbody>
<tr>
<td>Year 5</td>
<td>8</td>
<td>14</td>
<td>10</td>
</tr>
<tr>
<td>Year 6</td>
<td>7</td>
<td>9</td>
<td>5</td>
</tr>
</tbody>
</table>

One teacher looked at the data and said that going to the beach was more popular in Year 6 than Year 5.

**Do you agree with the teacher’s statement? YES or NO**

*Use mathematics to support your answer.*

---

8. **Number Line**

Use your ruler to **accurately** measure and mark the **exact** position of number 51 on the number line.

39 \[\boxed{\text{111}}\]

Explain how you located 51 at this spot.

---
Instructions:

1. Write the answer to each question in the space provided.
2. Use pen only.
3. Do not erase or white-out your answer. If you want to change your answer, cross out your answer and rewrite underneath.
4. Please explain your answers in as much detail as possible to show your thinking and solution strategy.
5. Calculators are not allowed.
6. Time allowed: 40 minutes.
7. Just do your best, even if you are finding some questions quite difficult. We are really interested in your thinking.
1. **Speedy Geoff**

Geoff runs 100 metres in 12 seconds.

If he runs the same distance at half the speed, how long will it take him?

**Answer:**

*Explain your thinking.*

2. **Balancing**

The picture below shows a balance beam with two different shaped objects hanging from it. With the objects arranged in this way, the beam is in balance.

Which object is heavier?

- [ ] The plain dark square
- [ ] The patterned square
- [ ] Both objects weigh the same
- [ ] You can’t tell

*Explain the reason for your choice.*
3. Washing Days

Which is better value, □ A or □ B? (tick to indicate your answer)

Explain your thinking.

4. Funky Music

Sandra wants to buy an MP3 player costing $210. Her mother agreed to pay $5 for every $2 Sandra saved.

How much did each person end up paying, once they had enough money?

Sandra pays: □ □

Mother pays: □ □

Explain your thinking.
5. **Cycling Home**

Anne was cycling home from school. She rode for a short time at a steady speed then stopped for a rest. When she started again, she rode twice as fast to get home quickly.

**Choose the graph that best represents her journey.**

- [ ] Graph A
- [ ] Graph B
- [ ] Graph C
- [ ] Graph D
- [ ] Graph E
- [ ] Graph F

**Give reasons for your choice referring to particular parts on the graph.**

____________________________________________________________________________________

____________________________________________________________________________________

____________________________________________________________________________________

____________________________________________________________________________________

____________________________________________________________________________________

____________________________________________________________________________________
6. Sinking or Floating

Tony has a collection of cubes and is testing to see whether they sink or float. He has created a table that shows the mass of each cube and its volume (below).

<table>
<thead>
<tr>
<th>Cube</th>
<th>Mass</th>
<th>Volume</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>10</td>
<td>100</td>
</tr>
<tr>
<td>B</td>
<td>20</td>
<td>90</td>
</tr>
<tr>
<td>C</td>
<td>30</td>
<td>80</td>
</tr>
<tr>
<td>D</td>
<td>40</td>
<td>70</td>
</tr>
<tr>
<td>E</td>
<td>50</td>
<td>60</td>
</tr>
<tr>
<td>F</td>
<td>60</td>
<td>50</td>
</tr>
</tbody>
</table>

He tests cube D ONLY and finds it sinks.

Study the table and indicate for each statement, whether it is TRUE or FALSE. Give a reason for your choice:

(a) TRUE or FALSE Cubes E and F will sink
Reason:

(b) TRUE or FALSE You can be sure cube C will sink
Reason:

(c) TRUE or FALSE Cube A is the most likely to float
Reason:
7. Juicy Drink

A fruit drink needs to be made up with one part of juice to four parts of water. You make enough drink for three glasses, each containing 200 mL.

How much juice would you need to make 600 mL of fruit drink?

Answer:_______ mL

Explain your thinking.

8. Tree Growth

Two palm trees were planted. One tree was 30 cm tall; the other tree was 90 cm tall. Both trees grow at the same rate.

When the shorter tree is 150 cm tall, what is the height of the taller tree?

Fill in the blank:
The height of the taller tree would be_______ cm

Explain your thinking.
Keeping it in Proportion
Part C

Time Allowed: 15 minutes

Equipment Required: Calculator

Instructions:
1. Write the answer to each question in the space provided.
2. Use pen only.
3. Do not erase or white-out your answer. If you want to change your answer, cross out your answer and rewrite underneath.
4. Please explain your answers in as much detail as possible to show your thinking and solution strategy.
5. Calculators allowed.
6. Time allowed: 15 minutes.
7. Just do your best, even if you are finding some questions quite difficult. We are really interested in your thinking.
Filling Up
Read the information given in the advertisement.
Petrol prices vary.
At what prices does 5% off beat 4c off per litre?

5% off
BP petrol
(beats 4c off per litre)

Use as much mathematics (including graphs if applicable) to explain your reasoning.
Continue overpage if you require more space.

The new BP-Citibank Mastercard